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$\mathscr{H}^{m,p}$-Extensions by $\mathscr{H}^{m,p}$-Splines

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1. INTRODUCTION

Let E be an arbitrary set of real numbers, f a real-valued function defined on E. We denote by $\mathscr{H}^{m,p} = \mathscr{H}^{m,p}(\mathbb{R})$ $(1 < p < \infty, m = 1, 2,...)$ the space of real-valued functions which are the m-fold integrals of functions in $\mathscr{L}^p = \mathscr{L}^p(\mathbb{R})$.[1] The main problems considered in this paper are the following:

PROBLEM I. Under what conditions is there a function $F \in \mathscr{H}^{m,p}$ such that $F(x) = f(x)$ for $x \in E$ (in other words, an $\mathscr{H}^{m,p}$-extension of f)?

PROBLEM II. To determine existence and uniqueness of an extremal (or minimal) $\mathscr{H}^{m,p}$-extension F_* for which $\int_{\mathbb{R}} | D^m F_* |^p$ is minimal.

PROBLEM III. To characterize F_* as the solution of a multipoint boundary-value problem.

We will also consider the $\mathscr{H}^{m,p}$-extension of a Taylor field on E. In this case we are given

$$f_0(x), f_1(x),..., f_{\mu(x)-1}(x), \qquad x \in E, \tag{1.1}$$

where $\mu(x)$, the *height* of the field at x, is a positive integer $\leqslant m$. We seek an $\mathscr{H}^{m,p}$-extension of the field, that is, a function $F \in \mathscr{H}^{m,p}$ such that

$$(1/k!)\, D^k F(x) = f_k(x), \qquad x \in E, \qquad k = 0, 1,..., \mu(x) - 1. \tag{1.2}$$

In 1934 [1] Hassler Whitney proposed and solved the following problem. Let E be a closed set, $E \subset I = [a, b]$, f a function $E \to \mathbb{R}$, and m a positive

[1] It should be observed that $F \in \mathscr{H}^{m,p}$ does not imply $F, DF,..., D^{m-1}F \in \mathscr{L}^p$.

integer. What are necessary and sufficient conditions that there exists a function $F \in \mathscr{C}^m[a, b]$ such that $F(x) = f(x)$ for $x \in E$? A necessary condition for the existence of such an extension is evidently the following. Let x' be any limit point of the set E, let $x_0, x_1, ..., x_m$ by distinct points of E and $f(x_0, x_1, ..., x_m)$ the m-th divided difference of f at these points. Then it is necessary that

$$\lim f(x_0, x_1, ..., x_m) \quad \text{exists as} \quad x_0, x_1, ..., x_m \to x' \text{ in } E \tag{1.3}$$

or, as Whitney expresses it, the divided differences of f of order m *converge on* E. He proved that this condition is also sufficient (for related results see [2–4]). The same condition is also necessary and sufficient for the $\mathscr{C}^m$-extendability of the Taylor field (1.1), defined on the closed set $E \subset I$, if the divided difference $f_0(x_0, x_1, ..., x_m)$ is understood as an *extended divided difference*. That is, if $\mu(x_0) > 1$ we may assume $x_0 = x_1 = \cdots = x_k$ for $k \leqslant \mu(x_0)$ and in the recursive definition of $f_0(x_0, x_1, ..., x_m)$ the divided difference $f_0(x_0, x_1, ..., x_k)$ is replaced by $f_k(x_0)$.

A slightly strengthened form of Whitney's theorem in which E may be any subset of $\mathbb{R}$, is proved in Golomb–Schoenberg [5]. Assume f is a continuous function $E \to \mathbb{R}$ (that is, the restriction of a continuous function $\mathbb{R} \to \mathbb{R}$ to E) and X is a dense subset of E, f_X the restriction of f to X. If f_X has an extension $F \in \mathscr{C}^m(\mathbb{R})$ then the restriction of F to E is f, hence F is also a $\mathscr{C}^m(\mathbb{R})$-extension of f. Therefore the strengthened version of Whitney's theorem gives

THEOREM (Whitney, Golomb–Schoenberg). *Suppose E is an arbitrary subset of $\mathbb{R}$, f a continuous function $E \to \mathbb{R}$, X a dense subset of E. f has an extension $F \in \mathscr{C}^m(\mathbb{R})$ if and only if the divided differences $f(x_0, x_1, ..., x_m)$ $(x_i \in X)$ converge. Similarly, a Taylor field* (1.1), *defined and continuous on E, has a $\mathscr{C}^m(\mathbb{R})$ extension if and only if the extended divided differences $f_0(x_0, x_1, ..., x_m)$ $(x_i \in X)$ converge.*

This version of Whitney's theorem is the analogue of Theorem 2.1 below. If $E = \mathbb{R}$ in the above theorem then $F = f$, and we obtain as a special case the

COROLLARY (Brouwer, Golomb–Schoenberg). *Suppose $f \in \mathscr{C}(\mathbb{R})$ and X is a dense subset of $\mathbb{R}$. f is in $\mathscr{C}^m(\mathbb{R})$ if and only if the divided differences $f(x_0, x_1, ..., x_m)$ $(x_i \in X)$ converge. Similarly, a Taylor field* (1.1), *defined and continuous on $\mathbb{R}$, is in $\mathscr{C}^m(\mathbb{R})$ if and only if the extended divided differences $f(x_0, x_1, ..., x_m)$ $(x_i \in X)$ converge.*

The last proposition does not deal with a problem of extension, but of characterization. It was established (for the special case $X = \mathbb{R}$), long before Whitney's theorem, by L. E. J. Brouwer in 1907 [6]. Similarly, there is an $\mathscr{H}^{m,p}(\mathbb{R})$-characterization theorem due to Riesz [7] for the case $m = 1$, p arbitrary; to Schoenberg [8] for the case $p = 2$, m arbitrary; and to Jerome–Schumaker [9] for the general case. A necessary and sufficient condition is again given in terms of the divided differences of f.

THEOREM (Riesz, Schoenberg, Jerome–Schumaker). *The function f is in $\mathscr{H}^{m,p}$ if and only if*

$$\sup \sum_{i=1}^{n-m} |f(x_i, x_{i+1}, \ldots, x_{i+m})|^p (x_{i+m} - x_i) < \infty \tag{1.4}$$

for arbitrary n-tuples ($n = m + 1, m + 2, \ldots$) $x_1 < x_2 < \cdots < x_n$ in $\mathbb{R}$.

It is also shown in [9] that if $f \in \mathscr{C}(\mathbb{R})$ then it is sufficient to require (1.4) only for equidistant n-tuples. This result is a special case of the $\mathscr{H}^{m,p}$-extension Theorem 3.3 below. Indeed, $f \in \mathscr{C}(\mathbb{R})$ is in $\mathscr{H}^{m,p}(\mathbb{R})$ if the restriction of f to some dense set X has an $\mathscr{H}^{m,p}$-extension.

2. First Solution of Problems I and II

Suppose E is a finite set

$$E = \{x_1, x_2, \ldots, x_n\}, \tag{2.1}$$

where we assume $n \geqslant m$. By [9, p. 364] the function $f\colon E \to \mathbb{R}$ has an $\mathscr{H}^{m,p}$-extension and, in particular, a unique extremal $\mathscr{H}^{m,p}$-extension, which we call the *$\mathscr{H}^{m,p}$-spline interpolating f* (called p-spline in [9]). A characterization of the extremal extension F_* as the solution of a multipoint boundary-value problem will be given in Section 4.

If we have a Taylor field $\{f_i\}$ [see (1.1)] defined on the finite set E, of order $\leqslant m - 1$ (i.e., $\mu(x_i) \leqslant m$, $x_i \in E$), then one concludes again that a unique extremal $\mathscr{H}^{m,p}$-extension exists, given by a function F_* which satisfies the interpolation conditions (1.2). The corresponding multipoint boundary-value problem is also given in Section 4.

A solution of Problem I of the Introduction may be given in terms of $\mathscr{H}^{m,p}$-splines which interpolate f on finite subsets of E.

THEOREM 2.1. *The function f with domain $E \subset \mathbb{R}$ has an $\mathscr{H}^{m,p}$-extension if and only if*

$$\sup_e \int_{\mathbb{R}} |D^m F_e|^p < \infty. \tag{2.2}$$

Here F_e denotes the $\mathscr{H}^{m,p}$-spline that interpolates f on the finite subset e of E, and the supremum is taken over the class of all finite subsets.

Proof. The proof is very similar to that of the corresponding theorem concerning $\mathscr{H}^{m,2}$-extension in [5]. Because F_e is the extremal $\mathscr{H}^{m,p}$-extension of the restriction of f to e, we have

$$\int_{\mathbb{R}} | D^m F_e |^p \leqslant \int_{\mathbb{R}} | D^m F |^p \tag{2.3}$$

for any $\mathscr{H}^{m,p}$-extension F of f. Thus, (2.2) is necessary.

Conversely, assume (2.2) is satisfied. Then one shows in the same way as in [5] that f is continuous. Let $\{x_1, x_2, ...\}$ be a sequence dense in E, and let e_n be the section $\{x_1, ..., x_n\}$. Because of the minimizing property of the spline $F_n = F_{e_n}$ the sequence $\{\int | D^m F_n |^p\}$ is monotone nondecreasing, and since it is bounded, the limit

$$\lim_{n\to\infty} \int_{\mathbb{R}} | D^m F_n |^p = L \tag{2.4}$$

exists. If $L = 0$ then $D^m F_n = 0$ for each n, $F_n \in \mathscr{P}^{m-1}$, hence $F_m = F_{m+1} = \cdots = F_*$ is an (extremal) $\mathscr{H}^{m,p}$-extension of f. If $L > 0$ then for arbitrary ϵ, $0 < \epsilon < 1$, there exists N_ϵ such that

$$1 - \epsilon \leqslant L^{-1} \int_{\mathbb{R}} | D^m F_n |^p \leqslant L^{-1} \int_{\mathbb{R}} | \tfrac{1}{2} D^m F_n + \tfrac{1}{2} D^m F_{n'} |^p, \qquad N_\epsilon \leqslant n < n', \tag{2.5}$$

since $\frac{1}{2}(F_n + F_{n'})$ interpolates f at the points of e_n. Because $\mathscr{L}^p$ is uniformly convex, this implies

$$\int_{\mathbb{R}} | D^m F_n - D^m F_{n'} |^p = O(\epsilon) \qquad \text{as} \quad \epsilon \to 0. \tag{2.6}$$

Thus $\{D^m F_n\}$ is a Cauchy sequence in $\mathscr{L}^p$. It is known that $\mathscr{H}^{m,p}$ with the norm

$$G \to \left\{ \sum_{i=1}^{m} | G(x_i)|^p + \int_{\mathbb{R}} | D^m G |^p \right\}^{1/p} \tag{2.7}$$

is a Banach space. The sequence $\{F_n\}$ is Cauchy in this space, hence converges to some $F_* \in \mathscr{H}^{m,p}$. It follows that

$$\lim_{n\to\infty} D^k F_n(x) = D^k F_*(x), \qquad k = 0, 1, ..., m - 1 \tag{2.8}$$

uniformly on bounded sets, hence $F_*(x_i) = F_n(x_i) = f(x_i)$ $(i = 1, 2,...)$, and since F_* and f are continuous,

$$F_*(x) = f(x), \qquad x \in E. \tag{2.9}$$

Thus, F_* is an $\mathscr{H}^{m,p}$-extension of f.

From the proof of Theorem 2.1 it is evident that condition (2.2) can be weakened for continuous functions.

COROLLARY 2.1. *Suppose* $\{x_1, x_2,...\}$ *is a dense subset of* E, $e_n = \{x_1,..., x_n\}$, $F_n = F_{e_n}$ *is the* $\mathscr{H}^{m,p}$*-spline that interpolates the continuous function* f *on* E. *Then* f *has an* $\mathscr{H}^{m,p}$*-extension if the sequence* $\{\int_{\mathbb{R}} | D^m F_n |^p\}$ *is bounded.*

Problem II of the Introduction has a simple answer.

THEOREM 2.2. *Suppose condition* (2.2) *of Theorem* 2.1 *is satisfied so that* f *has an* $\mathscr{H}^{m,p}$*-extension. Then* f *has a unique extremal extension* F_*, *and* F_* *is the limit in the normed space* $\mathscr{H}^{m,p}$ *of any sequence of* $\mathscr{H}^{m,p}$*-splines* Φ_n *that interpolate* f *on finite subsets* η_n *of* E, *if* $\bigcup_N \bigcap_{n \geqslant N} \eta_n$ *is dense in* E.

Proof. The $\mathscr{H}^{m,p}$-extension F_* constructed in the proof of Theorem 2.1 is clearly extremal. Suppose F is another extremal $\mathscr{H}^{m,p}$-extension. Then $\frac{1}{2}F + \frac{1}{2}F_*$ is also an extremal extension of f, thus

$$\left\{\int_{\mathbb{R}} | D^m F |^p\right\}^{1/p} + \left\{\int_{\mathbb{R}} | D^m F_* |^p\right\}^{1/p} = \left\{\int_{\mathbb{R}} | D^m F + D^m F_* |^p\right\}^{1/p} \tag{2.10}$$

and since $\mathscr{L}^p$ is strictly convex, this implies $F = F_*$. It remains to prove the second part of the theorem.

It is no restriction to assume

$$\bigcup_N \bigcap_{n \geqslant N} \eta_n = \{x_1, x_2,...\}. \tag{2.11}$$

Let e_n denote the section $\{x_1,..., x_n\}$ and F_n the spline that interpolates f on e_n. By the proof of Theorem 2.1 we have

$$\lim_{n\to\infty} F_n = F_* \tag{2.12}$$

in the normed space $\mathscr{H}^{m,p}$. We show next that

$$\lim_{n\to\infty} \int_{\mathbb{R}} | D^m \Phi_n |^p = \int_{\mathbb{R}} | D^m F_* |^p = L. \tag{2.13}$$

Clearly, $\int | D^m \Phi_n |^p \leqslant L$ for all n. The sequence $\{D^m \Phi_n\}$ is weakly compact in the normed space $\mathscr{H}^{m,p}$. If (2.13) was not true then there would be a subsequence $\{D^m\Phi_\nu\}$ that converges weakly, say to Φ_*, and such that

$$\int_{\mathbb{R}} | D^m\Phi_* |^p < L. \tag{2.14}$$

Since for each $i = 1, 2, ...,$ $\Phi_\nu(x_i) = f(x_i)$ for almost all ν, we have $\Phi_*(x_i) = f(x_i)$. Thus Φ_* is an $\mathscr{H}^{m,p}$-extension of f for which (2.14) holds, which is impossible. Therefore, (2.13) is proved.

Let Ψ_n be the $\mathscr{H}^{m,p}$-spline that interpolates f on $\eta_1 \cup \cdots \cup \eta_n$. Then $\Psi_n \to F_*$ in the normed space $\mathscr{H}^{m,p}$, and since $\frac{1}{2}(\Phi_n + \Psi_n)$ interpolates f on η_n, we have

$$\int_{\mathbb{R}} | D^m\Phi_n |^p \leqslant \int_{\mathbb{R}} | \tfrac{1}{2}D^m\Phi_n + \tfrac{1}{2}D^m\Psi_n |^p. \tag{2.15}$$

Proceeding with Φ_n and Ψ_n as with F_n and $F_{n'}$ in the proof of Theorem 2.1, we conclude

$$\lim_{n\to\infty} \int_{\mathbb{R}} | D^m\Phi_n - D^m\Psi_n |^p = 0, \tag{2.16}$$

hence $\Phi_n \to F_*$ in the normed space $\mathscr{H}^{m,p}$.

If we take the special case $E = \mathbb{R}$ in Theorem 2.2, we obtain a general result concerning the convergence of spline interpolants of $\mathscr{H}^{m,p}$-functions.

COROLLARY 2.2. *The function $f \in \mathscr{H}^{m,p}(\mathbb{R})$ is the limit in the normed space $\mathscr{H}^{m,p}$ of any sequence of $\mathscr{H}^{m,p}$-splines that interpolate f on finite sets η_n, if $\bigcup_N \bigcap_{n\geqslant N} \eta_n$ is dense in $\mathbb{R}$.*

It is readily seen that all the results of Section 2 are also valid for the Taylor field (1.1) in place of the function f.

3. Solution of Problems I and II in Special Cases

The numbers $\int | D^mF_e |^p$ are not easily calculated or even estimated, hence condition (2.2) is not practical as it stands. We establish other conditions for special cases.

Suppose E is an infinite monotone sequence

$$E\colon x_0 < x_1 < x_2 < \cdots \tag{3.1a}$$

or bisequence

$$E\colon \cdots < x_{-1} < x_0 < x_1 < \cdots \tag{3.1b}$$

and the conditions

$$\inf_{x_i \in E} (x_{i+1} - x_i) \geqslant h^{-1}, \qquad \sup_{x_i \in E} (x_{i+1} - x_i) \leqslant h, \tag{3.2}$$

are satisfied. We prove

THEOREM 3.1. *Let E be one of the sets* (3.1a) *or* (3.1b) *for which* (3.2) *holds and f a function $E \to \mathbb{R}$. There exists an $\mathscr{H}^{m,p}(\mathbb{R})$-extension of f if and only if*

$$\sum_{x_i \in E} |f(x_i, x_{i+1}, ..., x_{i+m})|^p < \infty. \tag{3.3}$$

The extremal $\mathscr{H}^{m,p}$-extension is $F_* = \lim F_n$ (convergence in the normed space $\mathscr{H}^{m,p}$), where F_n is the $\mathscr{H}^{m,p}$-spline interpolating f on $\{x_0, x_1, ..., x_n\}$ in case (3.1a), *and on $\{x_{-n}, ..., x_0, ..., x_n\}$ in case* (3.1b).

Proof. We give the proof only for the case (3.1a). The proof goes through for the other case with little change.

To prove the necessity of (3.3) suppose F is an $\mathscr{H}^{m,p}$-extension. Then

$$\begin{aligned} f(x_i, x_{i+1}, ..., x_{i+m}) &= F(x_i, x_{i+1}, ..., x_{i+m}) \\ &= (x_{i+m} - x_i)^{-1} [F(x_{i+1}, ..., x_{i+m}) - F(x_i, ..., x_{i+m-1})] \\ &= (x_{i+m} - x_i)^{-1} [D^{m-1}F(\eta_i) - D^{m-1}F(\xi_i)]/(m-1)!, \end{aligned} \tag{3.4}$$

where

$$x_i \leqslant \xi_i \leqslant x_{i+m-1}, \qquad x_{i+1} \leqslant \eta_i \leqslant x_{i+m}.$$

Hence, by the use of (3.2), (3.4) and Hölder's inequality,

$$\begin{aligned} \sum_{i=0}^{n} |f(x_i, ..., x_{i+m})|^p &\leqslant h \sum_{i=0}^{n} (x_{i+m} - x_i)^{-p+1} | D^{m-1}F(\eta_i) - D^{m-1}F(\xi_i)|^p \\ &\leqslant h \sum_{i=0}^{n} (\eta_i - \xi_i)^{-p+1} | D^{m-1}F(\eta_i) - D^{m-1}F(\xi_i)|^p \\ &\leqslant h \sum_{i=0}^{n} \int_{\xi_i}^{\eta_i} | D^m F |^p \\ &\leqslant mh \int_{\mathbb{R}} | D^m F |^p \end{aligned} \tag{3.5}$$

for all $n \geqslant m$, and the necessity of (3.3) is proved.

To prove the sufficiency of condition (3.3) we construct, for each $n \geqslant m$, a function $H_n \in \mathscr{H}^{m,p}$ such that

$$H_n(x_i) = f(x_i), \qquad i = 0, 1,..., n, \tag{3.6}$$

and

$$\int_{\mathbb{R}} | D^m H_n |^p \leqslant K \sum_{i=0}^{\infty} | f(x_i ,..., x_{i+m})|^p, \qquad n = m, \quad m + 1,... \tag{3.7}$$

for some constant K. Because of the minimizing property of the spline interpolant F_n we have $\int | D^m F_n |^p \leqslant \int | D^m H_n |^p$, hence by (3.7) and (3.4) the sequence $\{\int | D^m F_n |^p\}$ is bounded. By Theorem 3.1 we conclude that f has an $\mathscr{H}^{m,p}$-extension and that the extremal extension is $F_* = \lim F_n$.

To construct the functions H_n we introduce functions G_i ($i = 0, 1,...$) with the following properties:

$$\begin{aligned} &G_i \in \mathscr{C}^{m-1}(\mathbb{R}), \\ G_i(x_j) &= 0, \qquad j = i + 1, i + 2,..., i + m - 1, \\ G_i(x_j , x_{j+1} ,..., x_{j+m}) &= \delta_{ij} , \qquad j = 0, 1, 2,..., \\ G_i(x) &= 0, \qquad x < x_i , \\ D^{3m-1}G_i(x) &= 0, \qquad x_i \leqslant x < x_{i+m} , \\ D^m G_i(x) &= 0, \qquad x_{i+m} < x. \end{aligned} \tag{3.8}$$

We show below that there is a unique function G_i with these properties and that, furthermore, there is a constant C such that

$$\sup_{x\in\mathbb{R}} | D^m G_i(x)| \leqslant C, \qquad i = 0, 1, 2,... . \tag{3.9}$$

Then let $P \in \mathscr{P}^{m-1}$ be determined so that $P(x_j) = f(x_j)$ ($j = 0, 1,..., m - 1$) and define

$$H_n = P + \sum_{i=0}^{n-m} f(x_i , x_{i+1} ,..., x_{i+m})\, G_i , \qquad n = m, \quad m + 1,... . \tag{3.10}$$

By (3.8) we have

$$H_n(x_j) = P(x_j) = f(x_j) \qquad \text{for} \quad j = 0, 1,..., m - 1,$$

and

$$H_n(x_j ,..., x_{j+m}) = f(x_j ,..., x_{j+m}) \qquad \text{for} \quad j = 0, 1,..., n - m.$$

Therefore, (3.6) holds and since for $j = 0, 1, \ldots, n-1$

$$D^m H_n(x) = \sum_{i=\max(0,j-m+1)}^{\min(j,n-m)} f(x_i, \ldots, x_{i+m})\, D^m G_i(x), \qquad x_j \leqslant x \leqslant x_{j+1} \tag{3.11}$$

we have, by (3.9),

$$| D^m H_n(x) | \leqslant C \sum | f(x_i, \ldots, x_{i+m}) |;$$

hence

$$| D^m H_n(x) |^p \leqslant C_1 \sum | f(x_i, \ldots, x_{i+m}) |^p$$

and

$$\begin{aligned} \int_{\mathbb{R}} | D^m H_n |^p &= \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} | D^m H_n |^p \\ &\leqslant C_1 m h \sum_{i=0}^{n-m} | f(x_i, \ldots, x_{i+m}) |^p, \end{aligned} \tag{3.12}$$

which proves (3.7) with $K = C_1 m h$.

It remains to construct functions G_i with properties (3.8) and (3.9). We first determine, for $i = 0, 1, \ldots,$ numbers $y_{i,0}, \ldots, y_{i,m-1}$ such that

$$\begin{aligned} [x_i, x_{i+1}, \ldots, x_{i+m}; 0, 0, \ldots, 0, y_{i,0}] &= 1, \\ [x_{i+1}, x_{i+2}, \ldots, x_{i+m+1}; 0, 0, \ldots, y_{i,0}, y_{i,1}] &= 0, \\ &\cdots \\ [x_{i+m-1}, x_{i+m}, \ldots, x_{i+2m-1}; 0, y_{i,0}, \ldots, y_{i,m-1}] &= 0. \end{aligned} \tag{3.13}$$

Here $[x_0, \ldots, x_k; y_0, \ldots, y_k]$ denotes the k-th divided difference of a function which has the value y_i at x_i $(i = 0, 1, \ldots, k)$. Using the formula

$$\begin{aligned} \delta_{x_0 x_1 \cdots x_k} = {} & \Big[1 \Big/ \prod_{j \neq 0} (x_0 - x_j)\Big] \delta_{x_0} + \Big[1 \Big/ \prod_{j \neq 1} (x_1 - x_j)\Big] \delta_{x_1} \\ & + \cdots + \Big[1 \Big/ \prod_{j \neq k} (x_k - x_j)\Big] \delta_{x_k} \end{aligned} \tag{3.14}$$

for the divided difference operator, one sees how the numbers $y_{i,0}, y_{i,1}, \ldots, y_{i,m-1}$ are found successively so that (3.13) is satisfied. One also sees that because of (3.2) there exists a constant C_2 such that

$$| y_{i,j} | \leqslant C_2, \qquad i = 0, 1, \ldots; \quad j = 0, 1, \ldots, m-1. \tag{3.15}$$

We next determine a polynomial $p_i \in \mathscr{P}^{m-1}$ such that

$$p_i(x_{i+m+j}) = y_{ij}, \qquad j = 0, 1,..., m-1. \tag{3.16}$$

Using the fact that, for any interval I,

$$\sup_{x\in I} | p_i(x)| \leqslant \max_{j=0,...,m-1} | y_{ij} | \sup_{x\in I} \sum_{j=i+m}^{i+2m-1} \left| \prod_{\substack{k=i+m \\ k\neq j}}^{i+2m-1} \frac{x - x_k}{x_j - x_k} \right| \tag{3.17}$$

we conclude, using (3.2) and (3.15), that the sequence $\{| p_i(x_{i+m})|\}$ is bounded. More generally, there is a constant C_3 such that

$$| D^j p_i(x_{i+m})| \leqslant C_3, \qquad i = 0, 1,...; \quad j = 0, 1,..., m-1. \tag{3.18}$$

Finally we determine, for each $i = 0, 1,...,$ a polynomial $q_i \in \mathscr{P}^{3m-2}$ such that

$$\begin{aligned} D^j q_i(x_i) &= 0, \\ q_i(x_{i+j}) &= 0, \qquad j = 0, 1,..., m-1, \\ D^j q_i(x_{i+m}) &= D^j p_i(x_{i+m}). \end{aligned} \tag{3.19}$$

Clearly such a polynomial exists, for example, we may set

$$q_i(x) = (x - x_i)^m \prod_{j=1}^{m-1} (x - x_{i+j})\, Q_i(x), \tag{3.20}$$

where $Q_i \in \mathscr{P}^{m-1}$ is determined so that the last m conditions (3.19) are satisfied. Using (3.2) and (3.18) one concludes that there exists a constant C such that

$$\max_{x_i \leqslant x \leqslant x_{i+m}} | D^m q_i(x)| \leqslant C, \qquad i = 0, 1,... . \tag{3.21}$$

We are now ready to define the functions G_i :

$$\begin{aligned} G_i(x) &= 0 & x &< x_i, \\ &= q_i(x) & x_i &\leqslant x < x_{i+m}, \\ &= p_i(x) & x_{i+m} &\leqslant x. \end{aligned} \tag{3.22}$$

Then $G_i \in \mathscr{C}^{m-1}(\mathbb{R})$ [observe (3.19)], and G_i is seen to satisfy all the other conditions (3.8). Since $D^m G_i(x) \neq 0$ only for $x_i < x < x_{i+m}$, (3.9) is identical with (3.21). This concludes the proof of Theorem 3.1.

We now prove a theorem, similar to the preceding one, but concerned with the $\mathscr{H}^{m,p}$-extension of Taylor fields. We are given again a sequence like

(3.1a) or a bisequence like (3.1b) and we assume that condition (3.2) is satisfied. For each point $x \in E$ we are given $\mu(x) \leqslant m$ functions

$$f_0(x), f_1(x), \ldots, f_{\mu(x)-1}(x) \tag{3.23}$$

and we seek a function $F \in \mathscr{H}^{m,p}(\mathbb{R})$ which induces on E the field (3.23), that is,

$$(1/k!)\, D^k F(x) = f_k(x), \qquad x \in E, \quad k = 0, 1, \ldots, \mu(x) - 1. \tag{3.24}$$

To formulate the following theorem it is convenient to modify the notation.

The point $x \in E$ will be called a $\mu(x)$-fold node and it is given $\mu(x)$ different labels indexed successively. Thus we have an infinite sequence

$$E: x_0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \tag{3.25a}$$

or a bisequence

$$E: \cdots x_{-1} \leqslant x_0 \leqslant x_1 \leqslant \cdots \tag{3.25b}$$

with strings of equalities no longer than m. The function f defined on E, is now multivalued. If $x \in E$ is the initial element of a string of $\mu(x)$ equalities in (3.25) then there are $\mu(x)$ values $f_0(x), f_1(x), \ldots, f_{\mu(x)-1}(x)$ assigned to it. The function F on $\mathbb{R}$ extends (or interpolates) $\{f\}$ if

$$(1/k!)\, D^k F(x) = f_k(x), \qquad x \in E, \quad k = 0, 1, \ldots, \mu(x) - 1. \tag{3.26}$$

We define the extended difference $f(x_i, x_{i+1}, \ldots, x_{i+m})$ for the multivalued function f as follows. Suppose p_i is the unique polynomial of degree m for which

$$(1/k!)\, D^k p_i(x_j) = f_k(x_j), \qquad j = i, i+1, \ldots, i+m, \quad k = 0, 1, \ldots, \nu(x_j) - 1, \tag{3.27a}$$

where $\nu(x_j)$ is the number of times x_j appears in the sector $x_i \leqslant x_{i+1} \leqslant \cdots \leqslant x_{i+m}$ of E. Then we set

$$f(x_i, x_{i+1}, \ldots, x_{i+m}) = (1/m!)\, D^m p_i\,. \tag{3.27b}$$

If F is a function in $\mathscr{C}^{m-1}(\mathbb{R})$, the extended divided difference $F(x_i, x_{i+1}, \ldots, x_{i+m})$ is defined in analogous fashion. If $P_i \in \mathscr{P}^m$ is such that

$$D^k P_i(x_j) = D^k F(x_j), \qquad j = i, i+1, \ldots, i+m; \quad k = 0, 1, \ldots, \nu(x_j) - 1 \tag{3.28a}$$

then

$$F(x_i, x_{i+1}, \dots, x_{i+m}) = (1/m!)\, D^m P_i. \tag{3.28b}$$

$F(x_i, \dots, x_{i+m})$ can also be obtained as the limit of divided differences with distinct arguments $y_i, \dots, y_{i+m}$, by letting $\nu(x_j) - 1$ of the y's coalesce at x_j $(j = i, \dots, i + m)$.

THEOREM 3.2. *Let E be one of the sets* (3.25a) *or* (3.25b) *of multiple nodes* [*multiplicity* $\mu(x) \leqslant m$] *for which*

$$\inf_{x_i \in E, x_{i+1} \neq x_i} (x_{i+1} - x_i) \geqslant h^{-1}, \qquad \sup_{x_i \in E} (x_{i+1} - x_i) \leqslant h \tag{3.29}$$

holds and let the Taylor field $\{f_i\}$ of order $\leqslant m - 1$ be defined on E [i.e., the multivalued function f with values $f_0(x), f_1(x), \dots, f_{\mu(x)-1}(x)$ at $x \in E$]. There exists a function $F \in \mathscr{H}^{m,p}(\mathbb{R})$ that induces the Taylor field $\{f\}$ on E [i.e., $(1/k!)\, D^k F(x) = f_k(x)$ for $x \in E$, $k = 0, 1, \dots, \mu(x) - 1$] if and only if

$$\sum_{x_i \in E} |f(x_i, x_{i+1}, \dots, x_{i+m}|^p < \infty. \tag{3.30}$$

The extremal $\mathscr{H}^{m,p}$-extension is $F_* = \lim F_n$ (convergence in the normed space $\mathscr{H}^{m,p}$), where F_n is the $\mathscr{H}^{m,p}$-spline interpolating the Taylor field $\{f\}$ on $e_n = \{x_0, x_1, \dots, x_n)$ in case (3.25a), *and on $\{x_{-n}, \dots, x_0, \dots, x_n\}$ in case* (3.25b).

Remark. Condition (3.30), in form identical with (3.3), differs from the latter because of the presence of equalities among the arguments $x_i, \dots, x_{i+m}$. There are exactly $\mu(x_i)$ terms in (3.30) involving the point x_i. This condition was conjectured by Schoenberg [10] for the special case

$$p = 2, \quad E = \mathbb{Z}, \quad \mu(x) = \text{constant} = 2, \quad m \geqslant 2.$$

Proof. We prove Theorem 3.2 for the case where E is the sequence

$$0 = x_0 \leqslant x_1 \leqslant x_2 \leqslant \cdots; \qquad x_{i+1} - x_i = 0 \text{ or } 1. \tag{3.31}$$

The modifications needed in proving the general case are technical in nature and are well illustrated by the proof of Theorem 3.1.

To prove the necessity of (3.30) assume F is an $\mathscr{H}^{m,p}$-extension. Then it follows from the definition of the extended divided differences that

$$\begin{aligned} f(x_i, x_{i+1}, \dots, x_m) &= (x_{i+m} - x_i)^{-1} [F(x_{i+1}, \dots, x_{i+m}) - F(x_i, \dots, x_{i+m-1})] \\ &= (x_{i+m} - x_i)^{-1} [D^{m-1}F(\eta_i) - D^{m-1}F(\xi_i)]/(m-1)!, \end{aligned} \tag{3.32}$$

where

$$x_i \leqslant \xi_i \leqslant x_{i+m-1}, \qquad x_{i+1} \leqslant \eta_i \leqslant x_{i+m}$$

(observe that $x_{i+m} \neq x_i$). From here on one proceeds as in (3.5) and obtains (3.30).

We now assume (3.30) is satisfied. The arrays

$$x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{2m-1} \tag{3.33a}$$

and

$$x_i \leqslant x_{i+1} \leqslant \cdots \leqslant x_{i+2m-1} \tag{3.33b}$$

differ by a translation if and only if there is the same configuration of equalities and inequalities in (3.33a) as in (3.33b). There are no more than a finite number M of different configurations. Hence (3.33b) is congruent to one of M configurations (3.33a), which we label $X_1, \ldots, X_M$. Let us consider one of them

$$X: x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{2m-1}. \tag{3.34}$$

We construct a function G with the properties (compare (3.8))

$$\begin{aligned} G &\in \mathscr{C}^{m-1}(\mathbb{R}), && \\ G(x_j) &= 0, && j = 1, 2, \ldots, m-1, \\ G(x_j, x_{j+1}, \ldots, x_{j+m}) &= \delta_{0j}, && j = 0, 1, 2, \ldots, \\ G(x) &= 0, && x < 0, \\ D^{3m-1}G(x) &= 0, && 0 \leqslant x < x_m, \\ D^m G(x) &= 0, && x_m < x. \end{aligned} \tag{3.35}$$

We make the convention that if x_j appears ν_j times among $x_1, \ldots, x_{m-1}$ then the second equation above means $D^k G(x_j) = 0$ for $k = 0, 1, \ldots, \nu_j - 1$. Similarly, $G(x_j, \ldots, x_{j+m})$ is to be interpreted as an extended divided difference. To construct G, we first determine, as in (3.13), numbers $y_0, y_1, \ldots, y_{m-1}$ such that

$$\begin{aligned} [x_0, x_1, \ldots, x_m; 0, 0, \ldots, 0, y_0] &= 1, \\ [x_1, x_2, \ldots, x_{m+1}; 0, 0, \ldots, y_0, y_1] &= 0, \\ &\cdots \\ [x_{m-1}, x_m, \ldots, x_{2m-1}; 0, y_0, y_1, \ldots, y_{m-1}] &= 0. \end{aligned} \tag{3.36}$$

The divided differences in (3.36) should be interpreted as extended divided differences, as defined above. If, for example, $x_i < x_{i+1} = \cdots = x_m$, then $[x_0, x_1, \ldots, x_m; 0, 0, \ldots, 0, y_0]$ denotes the extended divided difference of a function φ for which $D^j\varphi(x_i) = 0$ for $j = 0, \ldots, m-i-1$ and $D^{m-i}\varphi(x_i) = y_0$.

We next determine $p \in \mathscr{P}^{m-1}$ such that

$$p(x_{m+j}) = y_j, \qquad j = 0, 1, \ldots, m-1. \tag{3.37}$$

For multiple nodes these equations should be interpreted as similar ones above. For example if $x_m = x_{m+1} = \cdots = x_{m+\mu-1}$ then the first μ equations (3.37) mean: $D^j p(x_m) = y_j$ for $j = 0, 1,..., \mu - 1$.

Finally we determine $q \in \mathscr{P}^{3m-2}$ such that

$$\begin{aligned} D^j q(0) &= 0, \\ q(x_j) &= 0, \qquad j = 0, 1,..., m-1, \\ D^j q(x_m) &= D^j p(x_m) \end{aligned} \tag{3.38}$$

with the same conventions concerning multiple nodes as above. In particular, if $x_i < x_{i+1} = \cdots = x_m$ then the last equation is understood to mean $D^{j+m-i-1} q(x_m) = D^{j+m-i-1} p(x_m)$.

We now define

$$\begin{aligned} G(x) &= 0, & x &< 0, \\ &= q(x), & 0 \leqslant x &< x_m, \\ &= p(x), & x_m &< x. \end{aligned} \tag{3.39}$$

Then G satisfies all the conditions (3.35). If we carry out the described construction for each of the distinct configurations $X_1,..., X_M$ (3.34), we obtain functions $G^1,..., G^M$. If the array $x_i \leqslant x_{i+1} \leqslant \cdots \leqslant x_{i+2m-1}$ is congruent to X_{i*} we set

$$G_i(x) = G^{i*}(x - x_i). \tag{3.40}$$

We then have by (3.35)

$$\begin{aligned} G_i(x_{i+j}) &= 0, \qquad j = 0, 1,..., m-1, \\ G_i(x_{i+j}, x_{i+j+1},..., x_{i+j+m}) &= \delta_{ij}, \qquad j = 0, 1, 2,... . \end{aligned} \tag{3.41}$$

Let now $P \in \mathscr{P}^{m-1}$ be determined so that $P(x_j) = f(x_j)$ $(j = 0, 1,..., m-1)$, with interpretation concerning multiple nodes as above, and set

$$H_n = P + \sum_{i=0}^{n-m} f(x_i, x_{i+1},..., x_{i+m})\, G_i, \qquad n = m, \quad m+1,... . \tag{3.42}$$

By (3.41) we have

$$H_n(x_j) = P(x_j) = f(x_j) \qquad (j = 0, 1,..., m-1)$$

and

$$H_n(x_j,..., x_{j+m}) = f(x_j,..., x_{j+m}),$$

hence

$$H_n(x_i) = f(x_i) \qquad i = 0, 1,..., n. \tag{3.43}$$

This means that, when restricted to the subset $\{x_0, \ldots, x_n\}$ of E, the function H_n induces the given Taylor field $\{f\}$.

By (3.35) and (3.42),

$$D^m H_n(x) = \sum_{i=\max(0,j-n+1)}^{\min(j,n-m)} f(x_i, \ldots, x_{i+m})\, D^m G_i(x), \qquad x_j \leqslant x \leqslant x_{j+1}. \quad (3.44)$$

Since each of the functions G_i is a translate of one of the functions $G^1, \ldots, G^M$, and since $D^m G^j$ has its support contained in $[0, m]$ it follows that the sequence

$$\sup_{x \in \mathbb{R}} | D^m G_i(x) |, \qquad i = 0, 1, 2, \ldots \quad (3.45)$$

is bounded. We can then proceed as in the proof of Theorem 3.1 and conclude that the sequence $\{\int | D^m H_n |^p\}$ is bounded. By Theorem 2.1, this proves the sufficiency of condition (3.30).

In the next theorem we deal with a set of the form

$$E = \bigcup_{n=1}^{\infty} e_n, \qquad e_n = \{x_1^n < x_2^n < \cdots < x_{N(n)}^n\}, \qquad 0 < x_i^n < 1. \quad (3.46)$$

We assume that each e_n includes the m points $x_1, \ldots, x_m$ and that

$$\lim_{n \to \infty} \max \{x_1^n, x_2^n - x_1^n, \ldots, x_{N(n)}^n - x_{N(n)-1}^n, 1 - x_{N(n)}^n\} = 0, \quad (3.47)$$

$$\min_{i=1,\ldots,N(n)-1} (x_{i+1}^n - x_i^n) / \max_{i=1,\ldots,N(n)-1} (x_{i+1}^n - x_i^n) \geqslant \delta > 0, \qquad n = 1, 2, \ldots. \quad (3.48)$$

We say, E is a family of quasiuniform partitions e_n, dense in $[0, 1]$. As before, let the function $f: E \to \mathbb{R}$ be given. Because of (3.47) f is densely defined in $I = [0, 1]$. If f has an $\mathscr{H}^{m,p}(\mathbb{R})$-extension then f is continuous and its uniquely defined continuous extension F to I is in $\mathscr{C}^{m-1}(I)$. Clearly, the extremal $\mathscr{H}^{m,p}$-extension F_* of f is given by

$$\begin{aligned} F_*(x) &= \sum_{k=0}^{m-1} \frac{1}{k!} D^k F(0)\, x^k, && x < 0, \\ &= F(x), && 0 \leqslant x \leqslant 1, \quad (3.49) \\ &= \sum_{k=0}^{m-1} \frac{1}{k!} D^k F(1)(x-1)^k, && 1 < x. \end{aligned}$$

Thus, f has an $\mathscr{H}^{m,p}$-extension if and only if $F \in \mathscr{H}^{m,p}(I)$.

THEOREM 3.3. *Let E be a family of quasiuniform partitions e_n, dense in $[0, 1]$, and f a continuous function $E \to \mathbb{R}$. The continuous extension F of f to $I = [0, 1]$ is in $\mathscr{H}^{m,p}(I)$ if and only if*

$$\sup_n \sum_{i=1}^{N(n)-m} |f(x_i^n, x_{i+1}^n, ..., x_{i+m}^n)|^p (x_{i+m}^n - x_i^n) < \infty. \tag{3.50}$$

The extremal $\mathscr{H}^{m,p}(\mathbb{R})$-extension F_* of f is given by (3.49) and also by $F_* = \lim F_n$ (convergence in the normed space $\mathscr{H}^{m,p}$), where F_n is the $\mathscr{H}^{m,p}$-spline that interpolates f on $\{x_1^n, ..., x_{N(n)}^n\}$.

Proof. The proof of the necessity of condition (3.50) is very much like that of condition (3.3), and is also contained in [9], hence omitted. To prove sufficiency we proceed as in the proof of Theorem 3.1 and construct functions $H_n \in \mathscr{H}^{m,p}(I)$ such that

$$\begin{gathered} H_n(x_i^n) = f(x_i^n), \qquad i = 1, ..., N(n) - m + 1 \\ \int_I | D^m H_n |^p \leqslant K \sum_{i=1}^{N(n)-2m+1} |f(x_i^n, ..., x_{i+m}^n)|^p (x_{i+m}^n - x_i^n), \quad n = 1, 2, ... \end{gathered} \tag{3.51}$$

for some constant K. Then $\int_{\mathbb{R}} | D^m F_n |^p \leqslant \int_I | D^m H_n |^p$ and it follows from (3.50) and (3.51) that the sequence $\{\int | D^m F_n |^p\}$ is bounded. It follows readily that the sequence $\{F_n\}$ is bounded and equicontinuous on I. Let $\{F_\nu(x)\}$ be a subsequence such that $\lim F_\nu(x) = F_0(x)$, $x \in I$. From the weak compactness of the sequence $\{F_n\}$ in $\mathscr{H}^{m,p}$ with norm (2.7) it follows that $F_0 \in \mathscr{H}^{m,p}(I)$. Since $F_\nu(x_i^\nu) = f(x_i^\nu)$ and since $\bigcup_{\nu, i=1,...,N(\nu)-m+1} \{x_i^\nu\}$ is dense in I we conclude $F_0 = F$, hence $F \in \mathscr{H}^{m,p}(I)$. That $\lim F_n = F_*$ (in $\mathscr{H}^{m,p}$) follows now in the same way as in the proof of Theorem 2.2.

To construct the functions H_n we determine, for $n = 1, 2, ...$ and $i = 1, 2, ..., N(n) - 2m + 1$ numbers $y_{i,0}^n, ..., y_{i,m-1}^n$ such that

$$\begin{aligned} [x_i^n, x_{i+1}^n, ..., x_{i+m}^n ; 0, 0, ..., 0, y_{i,0}^n] &= 1, \\ [x_{i+1}^n, x_{i+2}^n, ..., x_{i+m+1}^n ; 0, 0, ..., y_{i,0}^n, y_{i,1}^n] &= 0, \\ \cdots \\ [x_{i+m-1}^n, x_{i+m}^n, ..., x_{i+2m-1}^n ; 0, y_{i,0}^n, ..., y_{i,m-1}^n] &= 0. \end{aligned} \tag{3.52}$$

As in (3.15) one concludes, using (3.48), that there is a constant K_1 such that

$$| y_{i,j}^n | \leqslant K_1 h_n^{m-1}, \qquad n = 1, 2, ..., \quad i = 1, 2, ..., N(n) - 2m + 1, \quad j = 0, 1, ..., m - 1, \tag{3.53}$$

where we have set

$$h_n = \max_i (x_{i+1}^n - x_i^n). \tag{3.54}$$

Next we determine polynomials $p_i^n \in \mathscr{P}^{m-1}$ and $q_i^n \in \mathscr{P}^{3m-2}$ such that

$$p_i^n(x_{i+m+j}^n) = y_{ij}^n \tag{3.55a}$$

and

$$\begin{aligned} D^j q_i^n(x_i^n) &= 0, \qquad n = 1, 2,..., \\ q_i^n(x_{i+j}^n) &= 0, \qquad i = 1, 2,..., N(n) - 2m + 1, \\ D^j q_i^n(x_{i+m}^n) &= D^j p_i^n(x_{i+m}^n). \end{aligned} \tag{3.55b}$$

Using (3.53), we find that there is a constant K_2 such that

$$\max_{x_i^n \leqslant x \leqslant x_{i+m}^n} | D^m q_i^n(x)| \leqslant K_2 , \qquad n = 1, 2,..., \quad i = 1, 2,..., N(n) - 2m + 1. \tag{3.56}$$

Finally, let $P_n \in \mathscr{P}^{m-1}$ be such that $P^n(x_i^n) = f(x_i^n)$ $(i = 1,..., m)$ and

$$H_n = P_n + \sum_{i=1}^{N(n)-2m+1} f(x_i^n, x_{i+1}^n ,..., x_{i+m}^n) \, G_i^n, \tag{3.57a}$$

where

$$\begin{aligned} G_i^n(x) &= 0, \qquad & x &< x_i^n, \\ &= q_i^n(x), \qquad & x_i^n &\leqslant x < x_{i+m}^n , \\ &= p_i^n(x), \qquad & x_{i+m}^n &\leqslant x. \end{aligned} \tag{3.57b}$$

Then it is seen that $H_n(x_i^n) = f(x_i^n)$ for $i = 1,..., N(n) - m + 1$, $H_n \in \mathscr{H}^{m,p}(I)$, and because of (3.47) and (3.56)

$$\begin{aligned} \int_I | D^m H_n |^p &\leqslant K_3 h_n \sum_{i=1}^{N(n)-2m+1} | f(x_i^n,..., x_{i+n}^n)|^p \\ &\leqslant K \sum_{i=1}^{N(n)-2m+1} | f(x_i^n,..., x_{i+n}^n)|^p \, (x_{i+m}^n - x_i^n). \end{aligned} \tag{3.58}$$

for some constants K_3 , K. Thus H_n satisfies conditions (3.51) and the theorem is proved.

4. Extremal $\mathscr{H}^{m,p}$-Extensions as Solutions of Boundary-Value Problems

Let $f: E \to \mathbb{R}$ have an $\mathscr{H}^{m,p}$-extension, hence a unique extremal $\mathscr{H}^{m,p}$-extension F_*. Since f must necessarily be continuous f has a unique continuous extension $\bar{f}$ to the closure $\bar{E}$ of E, and F_* is also the extremal $\mathscr{H}^{m,p}$-extension of $\bar{f}$. Thus in this section, where we assume the extensibility of f, it is no restriction to assume that the domain E of f is closed. For abbreviation we set

$$\| F \|_p = \left(\int_{\mathbb{R}} | F |^p \right)^{1/p}.$$

We first show that extremal $\mathscr{H}^{m,p}$-interpolants are characterized by a nonlinear orthogonality property.

THEOREM 4.1. *A necessary and sufficient condition that the function $F \in \mathscr{H}^{m,p}(\mathbb{R})$ is the extremal $\mathscr{H}^{m,p}$-interpolant of a function on the closed set E is*

$$\int_{\mathbb{R}} | D^m F |^{p-1} \operatorname{sgn}(D^m F) \cdot D^m G = 0 \tag{4.1}$$

for every function $G \in \mathscr{H}^{m,p}(\mathbb{R})$ that vanishes on E.

Proof. Let $\mathscr{H}_0^{m,p}(E)$ denote the class of functions $G \in \mathscr{H}^{m,p}$ that vanish on E. Assume (4.1) holds for every $G \in \mathscr{H}_0^{m,p}(E)$. Then Hölder's inequality gives

$$\begin{aligned} \| D^m F \|_p^p &= \int_{\mathbb{R}} | D^m F |^{p-1} \operatorname{sgn}(D^m F)\, D^m F \\ &= \int_{\mathbb{R}} | D^m F |^{p-1} \operatorname{sgn}(D^m F)(D^m F + D^m G) \\ &\leqslant \int_{\mathbb{R}} | D^m F |^{p-1} \, | D^m F + D^m G | \\ &\leqslant \| D^m F \|_p^{p-1} \, \| D^m F + D^m G \|_p . \end{aligned}$$

Hence, if we set $F + G = H$,

$$\| D^m F \|_p \leqslant \| D^m H \|_p \tag{4.2}$$

and since this is true for every function $H \in \mathscr{H}^{m,p}$ which agrees with F on E, the sufficiency of (4.1) is proved.

Now assume, $G \in \mathscr{H}_0^{m,p}(E)$ and

$$\int_{\mathbb{R}} | D^mF |^{p-1} \operatorname{sgn}(D^mF)\, D^mG = A > 0. \tag{4.3}$$

Given $\epsilon_0 > 0$ we choose b so large, $I_b = [-b, b]$, such that for all $| \epsilon | < \epsilon_0$

$$\int_{\mathbb{R}\setminus I_b} | D^mF - \epsilon D^mG |^{p-1} | D^mG | < \tfrac{1}{8}A. \tag{4.4}$$

Then we choose $\delta > 0$ such that for any set $D \subset I_b$ with measure $| D | < \delta$ we have

$$\int_D | D^mF |^{p-1} | D^mG | < \tfrac{1}{4}A. \tag{4.5}$$

Next we find a closed subset $C \subset I_b$ of measure $| C | > 2b - \delta$ in which D^mF and D^mG are continuous (Lusin's theorem). We also assume that C is so chosen that $D^mF(x) \neq 0$ for $x \in C$. Let

$$\min_{x \in C} | D^mF(x)| = \mu. \tag{4.6}$$

Furthermore choose $\epsilon_1 < \epsilon_0$ so that

$$\max_{x \in C} | D^mG(x)| < \mu/\epsilon_1 . \tag{4.7}$$

Then for $| \epsilon | < \epsilon_1$

$$\begin{aligned} &\int_{I_b} | D^mF |^{p-1} \operatorname{sgn}(D^mF - \epsilon D^mG)\, D^mG \\ &\quad \geqslant \int_C | D^mF |^{p-1} \operatorname{sgn}(D^mF - \epsilon D^mG)\, D^mG - \tfrac{1}{4}A \\ &\quad = \int_C | D^mF |^{p-1} \operatorname{sgn}(D^mF)\, D^mG - \tfrac{1}{4}A \\ &\quad \geqslant A - \tfrac{1}{8}A - \tfrac{1}{4}A - \tfrac{1}{4}A = \tfrac{3}{8}A. \end{aligned} \tag{4.8}$$

Thus we can choose $\epsilon_2 < \epsilon_1$ such that for $| \epsilon | < \epsilon_2$

$$\int_{I_b} | D^mF - \epsilon D^mG |^{p-1} \operatorname{sgn}(D^mF - \epsilon D^mG)\, D^mG > \tfrac{1}{4}A. \tag{4.9}$$

Equation (4.9) together with (4.4) yields

$$\epsilon \int_{\mathbb{R}} | D^mF - \epsilon D^mG |^{p-1} \operatorname{sgn}(D^mF - \epsilon D^mG)\, D^mG > \tfrac{1}{8}\epsilon A, \qquad 0 < \epsilon < \epsilon_2 . \tag{4.10}$$

Therefore, using Hölder's inequality again,

$$\begin{aligned}
\| D^mF - \epsilon D^mG \|_p^p &= \int_{\mathbb{R}} D^mF \mid D^mF - \epsilon D^mG \mid^{p-1} \operatorname{sgn}(D^mF - \epsilon D^mG) \\
&\quad - \epsilon \int_{\mathbb{R}} D^mG \mid D^mF - \epsilon D^mG \mid^{p-1} \operatorname{sgn}(D^mF - \epsilon D^mG) \\
&< \int_{\mathbb{R}} D^mF \mid D^mF - \epsilon D^mG \mid^{p-1} \operatorname{sgn}(D^mF - \epsilon D^mG) \\
&\leqslant \int_{\mathbb{R}} \mid D^mF \mid \mid D^mF - \epsilon D^mG \mid^{p-1} \\
&\leqslant \| D^mF \|_p \| D^mF - \epsilon D^mG \|_p^{p-1}
\end{aligned}$$

and since (4.3) implies $\| D^mF \|_p > 0$,

$$\| D^mF - \epsilon D^mG \|_p < \| D^mF \|_p . \tag{4.11}$$

Since this is impossible, so is (4.3), and the necessity of condition (4.1) is proved.

We now set, for abbreviation,

$$\mid D^mF \mid^{p-1} \operatorname{sgn}(D^mF) = (D^mF)_s^{p-1}. \tag{4.12}$$

Let I be an open interval which contains none of the points of E. Then (4.1) implies

$$D^m(D^mF)_s^{p-1}(x) = 0, \qquad x \in I; \tag{4.13}$$

hence the restriction of $(D^mF)_s^{p-1}$ to I is a polynomial of degree $\leqslant m - 1$, say

$$(D^mF)_s^{p-1}(x) = P_I(x), \qquad x \in I. \tag{4.14}$$

Then

$$\begin{aligned}
\operatorname{sgn} D^mF(x) &= \operatorname{sgn} P_I(x), \\
\mid D^mF(x) \mid &= \mid P_I(x) \mid^{1/(p-1)}, \qquad x \in I, \\
D^mF(x) &= \mid P_I(x) \mid^{1/(p-1)} \operatorname{sgn} P_I(x).
\end{aligned} \tag{4.15}$$

In particular, D^mF is continuous in I and has one-sided limits at the finite endpoints of I. Also, D^mF is infinitely differentiable in I between zeros of P_I.

Suppose $a = \inf E > -\infty$. Since

$$\mid D^mF(x) \mid^p = \mid P_a(x) \mid^{1+[1/(p-1)]}, \qquad x < a, \tag{4.16}$$

where $P_a \in \mathscr{P}^{m-1}$, we conclude that F cannot be in $\mathscr{H}^{m,p}$ unless $P_a = 0$. The same conclusion holds for $x > b$ if $b = \sup E < \infty$. Thus

$$D^m F(x) = 0, \qquad x < \inf E \quad \text{and} \quad x > \sup E. \tag{4.17}$$

Now consider an isolated point x_i of E, and choose $\delta > 0$ so that $[x_i - \delta, x_i + \delta] \cap E = \{x_i\}$. Also choose functions $G_j \in \mathscr{C}^\infty(\mathbb{R})$ with support in $[x_i - \delta, x_i + \delta]$ and such that $D^k G_j(x_i) = \delta_{jk}$ ($j = 1,..., m - 1$; $k = 0,..., m - 1$). Then integration by parts in (4.1) gives

$$D^{m-1-j}(D^m F)_s^{p-1}(x_i + 0) - D^{m-1-j}(D^m F)_s^{p-1}(x_i + 0) = 0,$$
$$j = 1,..., m - 1. \tag{4.18}$$

Thus, $(D^m F)_s^{p-1}$ has continuous derivatives of order $\leqslant m - 2$ at the isolated points of E. In particular, if $a = \inf E > -\infty$ is an isolated point then by (4.17)

$$D^k(D^m F)_s^{p-1}(a + 0) = 0, \qquad k = 0, 1,..., m - 2 \tag{4.19}$$

and the corresponding result holds for $b = \sup E$.

In the following let E' denote the set of limit points of E. Both $\mathbb{R} \setminus E$ and $\mathbb{R} \setminus E'$ are open sets and are the unions of disjoint open intervals. Condition (4.13) can be expressed as $D^m(D^m F)_s^{p-1}(x) = 0$ for $x \in \mathbb{R} \setminus E$, and condition (4.18) as $(D^m F)_s^{p-1} \in \mathscr{C}^{m-2}(\mathbb{R} \setminus E')$. We thus have proved

THEOREM 4.2. *If F is the extremal $\mathscr{H}^{m,p}$-interpolant of a function defined on the closed set E then*

$$\begin{aligned} &\text{(i)} \quad (D^m F)_s^{p-1} \in \mathscr{C}^m(\mathbb{R} \setminus E) \cap \mathscr{C}^{m-2}(\mathbb{R} \setminus E'), \\ &\text{(ii)} \quad D^m(D^m F)_s^{p-1}(x) = 0,\ x \in \mathbb{R} \setminus E, \\ &\text{(iii)} \quad D^m F(x) = 0 \text{ for } x < \inf E \text{ and } x > \sup E. \end{aligned} \tag{4.20}$$

Condition (4.20i) reduces to a simpler one at points x where $D^m F(x) \neq 0$. Assume $D^m F(x) > 0$. Then

$$D(D^m F)_s^{p-1}(x) = (p - 1)(D^m F)^{p-2}(x)\, D^{m+1} F(x),$$

and this function is continuous at x if and only if $D^{m+1}F$ is. Similarly for the higher derivatives. Thus, we have

COROLLARY 4.1. *At points x where $D^m F(x) \neq 0$ condition* (4.20i) *of Theorem* 4.2 *is equivalent to*

The derivatives $DF(x),..., D^{2m}F(x)$ exist and are continuous if $x \notin E$. (4.20i′)
The derivatives $DF(x),..., D^{2m-2}F(x)$ exist and are continuous if $x \notin E'$.

We indicate the few changes that must be made if we deal with the extension of a Taylor field rather than a function. We assume E is closed, the Taylor field (1.1) is of order $\leqslant m-1$ and it is induced by a function in $\mathscr{H}^{m,p}(\mathbb{R})$, so that a function F with minimal $\int |D^mF|^p$ exists for which

$$(1/k!)\, D^kF(x) = f_k(x) \qquad k = 0, 1,..., \mu(x) - 1, \quad x \in E.$$

We say that F is the extremal $\mathscr{H}^{m,p}$-interpolant of a Taylor field on E of height μ. In place of Theorem 4.1 we have

THEOREM 4.1a. *A necessary and sufficient condition that the function* $F \in \mathscr{H}^{m,p}(\mathbb{R})$ *is the extremal* $\mathscr{H}^{m,p}$*-interpolant of a Taylor field of variable height* $\mu \leqslant m$ *on the closed set* E *is*

$$\int_{\mathbb{R}} |D^mF|^{p-1} \operatorname{sgn}(D^mF) \cdot D^mG = 0 \tag{4.21}$$

for every function $G \in \mathscr{H}^{m,p}(\mathbb{R})$ *for which*

$$D^kG(x) = 0, \qquad k = 0, 1,..., \mu(x) - 1, \quad x \in E. \tag{4.22}$$

Theorem 4.2 is replaced by

THEOREM 4.2a. *If* F *is the extremal* $\mathscr{H}^{m,p}$*-interpolant of a Taylor field of variable height* $\mu \leqslant m$ *on the closed set* E *then*

(i) $(D^mF)_s^{p-1} \in \mathscr{C}^m(\mathbb{R} \setminus E)$; $D^k(D^mF)_s^{p-1}(x)$ *exists and is continuous at* $x \in \mathbb{R} \setminus E'$ *for* $k = 0, 1,..., m - 1 - \mu(x)$ [*no condition if* $\mu(x) = m$],

(ii) $D^m(D^mF)_s^{p-1}(x) = 0$, $x \in \mathbb{R} \setminus E$, (4.23)

(iii) $D^mF(x) = 0$ *for* $x < \inf E$ *and* $x > \sup E$.

Corollary 4.1 also has its analogue. For x such that $D^mF(x) \neq 0$ condition (4.23i) is equivalent to

$DF(x),..., D^{2m}F(x)$ exist and are continuous if $x \notin E$.

$DF(x),..., D^{2m-1-\mu(x)}F(x)$ exist and are continuous if $x \notin E'$ (4.23i')

[no condition if $\mu(x) = m$].

We have shown that the extremal $\mathscr{H}^{m,p}$-interpolant of a function or a Taylor field on a closed set E is the solution of a multipoint boundary-value problem of the nonlinear (if $p \neq 2$) differential equation

$$D^m(D^mF)_s^{p-1} = 0, \tag{4.24}$$

where E is the set of boundary points, which appear as *knots* (points of diminished smoothness) of the solution. If x is a boundary point such that $x \in E \setminus E'$ and $\mu(x_i) < m$ then the condition

$$D^{m-i-\mu(x)}(D^m F)_s^{p-1}(x) \qquad \text{exists and is continuous,} \tag{4.25}$$

is nonvacuous, and specifies a *"spline continuation"* across the point x. If $x \in E'$ or $\mu(x) = m$, then there is no spline continuation across x, and we say these points form the *essential boundary* E_b of the problem

$$E_b = E' \bigcup \{x \in E: \mu(x) = m\}. \tag{4.26}$$

Suppose $x_* \in E_b$. Let $\{f^-\}$ and $\{f^+\}$ denote the restrictions of the Taylor field $\{f\}$ to $\{x \leqslant x_*\}$ and $\{x \geqslant x_*\}$, respectively, and let F_*^+ and F_*^- denote the corresponding extremal $\mathscr{H}^{m,p}$-interpolants. Then clearly

$$F_*(x) = \begin{cases} F_*^-(x), & x \leqslant x_*, \\ F_*^+(x), & x \geqslant x_*, \end{cases}$$

defines the extremal $\mathscr{H}^{m,p}$-interpolant of $\{f\}$. Thus each point of the essential boundary breaks the extremal extension problem up into uncoupled problems. E_b is a closed set. Let

$$\mathbb{R} \setminus E_b = \bigcup_\nu J_\nu, \tag{4.27}$$

where each J_ν is an open interval and $J_\mu \cap J_\nu = \phi$ for $\mu \neq \nu$. We call the J_ν the *disjoint intervals* of the extension problem. The above discussion shows that the extremal $\mathscr{H}^{m,p}$-extension problem breaks up into a set of uncoupled problems, one for each disjoint interval.

Let J be one of the disjoint intervals. The set $J \cap E$ is discrete and, moreover, $\mu(x) \leqslant m - 1$ for $x \in J \cap E$. The restriction of $\{f\}$ to $\overline{J \cap E}$ consists of

$$f_0(x), f_1(x), \ldots, f_{\mu(x)}(x), \qquad x \in J \cap E \tag{4.28a}$$

and

$$f_0(x_*), f_1(x_*), \ldots, f_{m-1}(x_*) \tag{4.28b}$$

at each of the finite endpoints of J (if any). Observe that $J \cap E$ may be empty. This can happen only if J has at least one finite endpoint since we assumed $\sum_{x \in E} \mu(x) \geqslant m$.

Put $E_J = J \cap E$ and suppose F_J is the restriction to $\bar{J}$ of the extremal $\mathscr{H}^{m,p}(\mathbb{R})$-interpolant of $\{f\}$.

By Theorem 4.2a we have

$$\begin{aligned}
&\text{(i)}\quad F_J \in \mathscr{H}^{m,p}(J);\ (D^m F_J)_s^{p-1} \in \mathscr{C}^m(\bar{J} \setminus \bar{E}_J);\ D^k (D^m F_J)_{s(x)}^{p-1} \text{ exists and is continuous at } x \in E_J \text{ for } k = 0, 1,..., m-1-\mu(x),\\
&\text{(ii)}\quad D^m (D^m F_J)_s^{p-1}(x) = 0,\ x \in J \setminus E_J,\\
&\text{(iii)}\quad (1/k!)\, D^k F_J(x) = f_k(x) \text{ for } k = 0, 1,..., \mu(x) - 1,\ x \in E_J;\ (1/k!)\, D^k F_J(x) = f_k(x) \text{ for } 0, 1,..., m-1,\ x \in \bar{E}_J \setminus E_J.
\end{aligned} \tag{4.29}$$

Observe that the last condition refers to the finite endpoints of E_J (if any). If $\sup J = \infty$, $\sup E_J < \infty$ then $D^m F_J(x) = 0$ for $x > \sup E_J$ by Theorem 4.2a. This condition is not formulated in (4.29) since it is an automatic consequence of (4.29ii) and the fact that $F_J \in \mathscr{H}^{m,p}(J)$. A similar comment holds for the case $\inf J = -\infty$, $\inf E_J > -\infty$.

Suppose now that the function F satisfies conditions (4.29). If E_J is a finite set then we can perform integration by parts in the integral

$$\int_J (D^m F)_s^{p-1}\, D^m G \tag{4.30}$$

and, using properties (4.29i, ii), we obtain 0 for any function $G \in \mathscr{H}^{m,p}(J)$ for which $D^k G(x) = 0$ $(k = 0, 1,..., \mu(x) - 1, x \in E_J)$ and $D^k G(x) = 0$ $(k = 0, 1,..., m-1; x \in \bar{E}_J \setminus E_J)$. By Theorem 4.1a and because of (4.29iii) F is the restriction to $\bar{J}$ of the extremal $\mathscr{H}^{m,p}(\mathbb{R})$-interpolant of $\{f\}$. Thus, we have proved

THEOREM 4.3. *If J is one of the disjoint intervals of the $\mathscr{H}^{m,p}$-extension problem and $E_J = J \cap E$ is finite then $F_J \in \mathscr{H}^{m,p}(J)$ is the restriction to J of the extremal $\mathscr{H}^{m,p}(\mathbb{R})$-interpolant of the Taylor field $\{f\}$ if and only if conditions* (4.29i, ii, iii) *are satisfied.*

Thus, for each disjoint interval J that contains only finitely many of the points of E, the component F_J of the extremal $\mathscr{H}^{m,p}(\mathbb{R})$-extension of $\{f\}$ is completely characterized as the solution of a boundary-value problem, as formulated in (4.29). In particular, if E is finite itself and $\mu(x) < m$ for all $x \in E$, (hence there is only one disjoint interval $J = \mathbb{R}$) then the extremal $\mathscr{H}^{m,p}(\mathbb{R})$-interpolant F_* of $\{f\}$ is completely characterized by (4.29).

In the next section this characterization of the extremal $\mathscr{H}^{m,p}$-interpolants will be proved for disjoint intervals containing infinitely many points of E. It should be observed that proving the sufficiency of conditions (4.29) is equivalent to proving that the boundary-value problem has a unique solution. In fact, if F_J is the extremal extension, then F_J is a solution of the boundary-value problem (4.29) by Theorem 4.2a, and if the latter has only one solution it must be F_J. Conversely, if every solution of (4.29) is an extremal extension then there can be no more than one solution since the extremal solution is unique.

5. Infinite Discrete Sets

As in the preceding section let J denote one of the disjoint intervals for the $\mathscr{H}^{m,p}$-extension problem. We now assume that the set $E_J = E \cap J$ of interpolation nodes contained in J is infinite.

To prove the following theorem we need a lemma.

Lemma 5.1. *Suppose* $x_0 < x_1 < x_2 < \cdots$, $\lim x_n = x_\infty \leqslant \infty$, $J = (x_0, x_\infty)$, $D^m U(x) = 0$ *for* $x \neq x_0, x_1, \ldots$, $\int_J |U|^q < \infty$ *for some* q, $1 < q < \infty$. *Then*

$$\sup_{x_n < x < x_{n+1}} |D^k U(x)| = o(|x_{n+1} - x_n|^{-k-1/q}) \quad \textit{as } n \to \infty \quad k = 0, 1, 2, \ldots. \tag{5.1}$$

Proof. Let

$$U(x) = \sum_{l=0}^{m-1} a_{nl} \left(\frac{x - x_n}{x_{n+1} - x_n}\right)^l, \qquad x_n < x < x_{n+1}. \tag{5.2}$$

Then

$$\frac{1}{x_{n+1} - x_n} \int_{x_n}^{x_{n+1}} |U(x)|^q \, dx = \int_0^1 \left| \sum_{l=0}^{m-1} a_{nl} x^l \right|^q dx \geqslant K \sum_{l=0}^{m-1} |a_{nl}|^q \tag{5.3}$$

for some $K > 0$ independent of n. Since the left-hand side is $o(|x_{n+1} - x_n|^{-1})$ as $n \to \infty$ we conclude

$$a_{nl} = o(|x_{n+1} - x_n|^{-1/q}) \qquad \text{as} \quad n \to \infty, \qquad l = 0, 1, \ldots, m-1. \tag{5.4}$$

From (5.2)

$$D^k U(x) = (x_{n+1} - x_n)^{-k} \sum_{l=k}^{m-1} a_{nl} \frac{l!}{(l-k)!} \left(\frac{x - x_n}{x_{n+1} - x_n}\right)^{l-k}, \qquad x_n < x < x_{n+1}$$

and by (5.4)

$$\begin{aligned} \sup_{x_n < x < x_{n+1}} |D^k U(x)| &\leqslant |x_{n+1} - x_n|^{-k} \sum_{l=k}^{m-1} |a_{nl}| \frac{l!}{(l-k!)} \\ &= o(|x_{n+1} - x_n|^{-k-1/q}) \qquad \text{as} \quad n \to \infty, \ k = 0, 1, 2, \ldots. \end{aligned} \tag{5.5}$$

Remark. The lemma is clearly valid also for $q = 1$ and $q = \infty$.
We now prove

THEOREM 5.1. *Suppose $J = (a, b)$, $-\infty \leqslant a < b \leqslant \infty$, E_J is a discrete set in J, and $\{f\}$ is the Taylor field*

$$\begin{aligned} &f_k(x), && k = 0, 1,..., \mu(x) - 1, && x \in E_J, \\ &f_k(a), && k = 0, 1,..., m - 1 && \text{if } a > -\infty, \\ &f_k(b), && k = 0, 1,..., m - 1 && \text{if } b < +\infty, \end{aligned} \tag{5.6}$$

where $\mu(x) \leqslant m - 1$, which is assumed to have an $\mathscr{H}^{m,p}(J)$ extension. Then the multipoint boundary-value problem

(i) $F \in \mathscr{H}^{m,p}(J)$, $(D^mF)_s^{p-1} \in \mathscr{C}^m(J \setminus E_J)$; $D^k(D^mF)_{s(x)}^{p-1}$ *exists and is continuous at $x \in E_J$ for $k = 0, 1,..., m - 1 - \mu(x)$.*

$$\begin{aligned} &\text{(ii)} \quad D^m(D^mF)_s^{p-1}(x) = 0, \ x \in J \setminus E_J. \\ &\text{(iii)} \quad (1/k!)D^kF(x) = f_k(x), \ k = 0, 1,..., \mu(x) - 1, \ x \in E_J, \\ &\qquad (1/k!)D^kF(a) = f_k(a), \ k = 0, 1,..., m - 1 \ \text{if } a > -\infty, \\ &\qquad (1/k!)D^kF(b) = f_k(b), \ k = 0, 1,..., m - 1 \ \text{if } b < +\infty, \end{aligned} \tag{5.7}$$

has a unique solution $F = F_J$. F_J is the extremal $\mathscr{H}^{m,p}(J)$-interpolant of the field $\{f\}$.

Proof. By Theorem 4.1a and the remarks at the end of Section 4 it suffices to prove

$$\int_J (D^mF_J)_s^{p-1} D^mG = 0 \tag{5.8}$$

for every function $G \in \mathscr{H}^{m,p}(J)$ for which

$$\begin{aligned} &D^kG(x) = 0, && k = 0, 1,..., \mu(x) - 1, && x \in E_J, \\ &D^kG(a) = 0, && k = 0, 1,..., m - 1 && \text{if } a > -\infty, \\ &D^kG(b) = 0, && k = 0, 1,..., m - 1 && \text{if } b < +\infty. \end{aligned} \tag{5.9}$$

We carry out the proof for the case

$$\begin{aligned} a > -\infty, \qquad b = \lim_{n \to \infty} x_n \leqslant \infty, \\ E_J = \{x_0 < x_1 < x_2 < \cdots\}. \end{aligned} \tag{5.10}$$

We set

$$U = (D^mF_J)_s^{p-1}. \tag{5.11}$$

Since $F_J \in \mathscr{H}^{m,p}(J)$, U is in $\mathscr{L}^q(J)$ $(p^{-1} + q^{-1} = 1)$, and because of (5.7ii), U satisfies the hypotheses of Lemma 7.1. Therefore,

$$\sup_{x_n < x < x_{n+1}} | D^k U(x)| = o(| x_{n+1} - x_n |^{-k-1/q}) \qquad \text{as} \quad n \to \infty,\ k = 0, 1, 2,... . \tag{5.12}$$

If one performs $(m - 1)$ integration by parts in (5.8) and uses (5.7i, ii) and (5.9), one obtains

$$\int_a^{x_n} U(x)\, D^m G(x)\, dx = \sum_{k=0}^{\mu(x_n)-1} (-1)^{k-1} (D^k U\, D^{m-k-1} G)(x_n). \tag{5.13}$$

Since $G(x_n) = 0$ $(n = 0, 1,...)$, there are points ξ_{nk} such that

$$\begin{aligned} x_n \leqslant \xi_{nk} \leqslant x_{n+m-1} \qquad n = 0, 1,..., \\ D^k G(\xi_{nk}) = 0 \qquad k = 1, 2,..., m - 1. \end{aligned} \tag{5.14}$$

Writing

$$D^{m-1} G(x) = \int_{\xi_{n,m-1}}^{x} D^m G(\xi)\, d\xi, \tag{5.15}$$

we obtain by the use of Hölder's inequality

$$\max_{x_n \leqslant x \leqslant x_{n+m-1}} | D^{m-1} G(x)| \leqslant | x_{n+m-1} - x |^{1/q} \left\{ \int_{x_n}^{x_{n+m-1}} | D^m G(\xi)|^p\, d\xi \right\}^{1/p};$$

hence

$$\max_{x_n \leqslant x \leqslant x_{n+m-1}} | D^{m-1} G(x)| = o(| x_{n+m-1} - x_n |^{1/q}) \qquad \text{as} \quad n \to \infty. \tag{5.16}$$

Since

$$D^{m-2} G(x) = \int_{\xi_{n,m-2}}^{x} D^{m-1} G(\xi)\, d\xi,$$

(5.16) gives

$$\max_{x_n \leqslant x \leqslant x_{n+m-1}} | D^{m-2} G(x)| = o(| x_{n+m-1} - x_n |^{1+1/q}) \qquad \text{as} \quad n \to \infty. \tag{5.17}$$

Proceeding in this fashion, one obtains generally

$$\max_{x_n \leqslant x \leqslant x_{n+m-1}} | D^{m-k-1} G(x)| = o(| x_{n+m-1} - x_n |^{k+1/q}) \qquad \text{as} \quad n \to \infty, \quad k = 0, 1,..., m - 1. \tag{5.18}$$

Set

$$\begin{aligned}\delta_n &= \max_{i=1,\ldots,m-1} (x_{n+i} - x_{n+i-1}) \\ &= x_{N(n)+1} - x_{N(n)}, \qquad n = 0, 1, 2,\ldots,\end{aligned} \tag{5.19a}$$

where

$$n \leqslant N(n) \leqslant n + m - 2. \tag{5.19b}$$

By (5.12),

$$D^k U(x_{N(n)}) = o(\delta_n^{-k-1/q}), \qquad k = 0, 1,\ldots$$

and by (5.18),

$$D^{m-k-1} G(x_{N(n)}) = o(\delta_n^{k+1/q}), \qquad k = 0, 1,\ldots .$$

Thus, (5.13) gives

$$\int_a^{x_{N(n)}} U(x)\, D^m G(x)\, dx = o(1) \qquad \text{as} \quad n \to \infty \tag{5.20}$$

and this proves (5.8).

If

$$\begin{aligned} a = \lim_{n\to\infty} x_{-n} \geqslant -\infty, &\qquad b = \lim_{n\to\infty} x_n \leqslant \infty, \\ E_J = \{\cdots x_{-2} < x_{-1} &< x_0 < x_1 < \cdots\},\end{aligned} \tag{5.21}$$

then we start with

$$\begin{aligned}\int_{x_{-n}}^{x_n} UD^m G = &\sum_{k=0}^{\mu(x_n)-1} (-1)^{k-1} (D^k U D^{m-k-1} G)(x_n) \\ &- \sum_{k=0}^{\mu(x_n)-1} (-1)^{k-1} (D^k U D^{m-k-1} G)(x_{-n})\end{aligned} \tag{5.22}$$

and proceed as above with each of the two sums in (5.22) separately, to conclude (5.8). This exhausts all possibilities, hence Theorem 5.1 is completely proved.

By the result of Theorem 5.1 and the remarks made in Section 4 concerning the breakup of the boundary-value problem (4.23) we have demonstrated that the boundary-value problem uniquely characterizes the extremal $\mathscr{H}^{m,p}$-extension of a function or Taylor field. This is the solution of Problem III of the Introduction.

6. The Cone of $\mathscr{H}^{m,p}$-Splines

We consider the class of extremal $\mathscr{H}^{m,p}$-interpolants of Taylor fields of given height $\mu \leqslant m$ on a given set E.

A function $F: \mathbb{R} \to \mathbb{R}$ that satisfies the conditions

$$\begin{aligned} &\text{(i)} \quad F \in \mathscr{H}^{m,p}(\mathbb{R}),\ (D^m F)_s^{p-1} \in \mathscr{C}^m(\mathbb{R} \setminus E) \cap \mathscr{C}^{m-2}(\mathbb{R} \setminus E'), \\ &\text{(ii)} \quad D^m (D^m F)_s^{p-1}(x) = 0,\ x \in \mathbb{R} \setminus E, \end{aligned} \tag{6.1}$$

where E is a closed subset of $\mathbb{R}$, will be called an *$\mathscr{H}^{m,p}$-spline with* (simple) *knots in E*. If F satisfies the conditions

(i) $F \in \mathscr{H}^{m,p}(\mathbb{R})$, $(D^m F)_s^{p-1} \in \mathscr{C}^m(\mathbb{R} \setminus E)$, $D^k (D^m F)_{s(x)}^{p-1}$ exists and is continuous at $x \in \mathbb{R} \setminus E'$ for $k = 0, 1,..., m - 1 - \mu(x)$, (6.2)

(ii) $D^m (D^m F)_s^{p-1}(x) = 0,\ x \in \mathbb{R} \setminus E$,

where E is a closed set and μ a function $E \to \{1,..., m - 1\}$, F will be called an *$\mathscr{H}^{m,p}$-spline with knots of* (variable) *multiplicity μ in E*.

It was seen in Sections 4 and 5 that F is an $\mathscr{H}^{m,p}$-spline with knots of multiplicities μ in E if and only if F is the extremal $\mathscr{H}^{m,p}$-interpolant of the Taylor field $\{f\}$

$$f_k(x) = (1/k!)\, D^k F(x), \qquad k = 0, 1,..., \mu(x) - 1, \quad x \in E. \tag{6.3}$$

It was also proved that F is an $\mathscr{H}^{m,p}$-spline with knots of multiplicity μ in E if and only if

$$\int_{\mathbb{R}} (D^m F)_s^{p-1}\, D^m G = 0 \tag{6.4a}$$

for every function G such that

$$D^k G(x) = 0, \qquad k = 0, 1,..., \mu(x) - 1, \quad x \in E. \tag{6.4b}$$

We denote the class of $\mathscr{H}^{m,p}$-splines with knots of multiplicity μ in E by $\mathscr{S} = \mathscr{S}_{E,\mu}^{m,p}$ ($\mathscr{S}_E^{m,p}$ if $\mu = 1$). We wish to study some topological properties of $\mathscr{S}$ as a subset of the Banach space $\mathscr{H}^{m,p}$ with the norm

$$G \to \| G \|_p = \left\{ \sum_{i=1}^{m} | G(x^i)|^p + \int_{\mathbb{R}} | D^m G |^p \right\}^{1/p}. \tag{6.5}$$

Here we will assume that $\{x^1,..., x^m\}$ is a fixed subset of E.

For a closed set E and multiplicity function $\mu: E \to \{1,..., m\}$ we have the essential boundary

$$E_b = E' \bigcup \{x \in E: \mu(x) = m\} \tag{6.6}$$

introduced in Section 4. We redefine, if necessary, the multiplicity function μ such that

$$\mu(x) = m, \qquad x \in E'. \tag{6.7}$$

This is justified since if the Taylor field $\{f\}$ has an $\mathscr{H}^{m,p}$-extension F, then $F \in \mathscr{C}^{m-1}(\mathbb{R})$ and $f \in \mathscr{C}^{m-1}(E)$ (by Whitney's definition). The values $f(x)$, $x \in E$, determine uniquely $f_1(x),..., f_{m-1}(x)$ for $x \in E'$ such that

$$(1/k!)\, D^k F(x) = f_k(x), \qquad k = 0, 1,..., m-1, \quad x \in E.$$

With convention (6.7), (6.6) becomes

$$E_b = \{x \in E: \mu(x) = m\}. \tag{6.8}$$

We call the points of E_b the *essential knots* and those of $E \setminus E_b$ the *non-essential knots* of the splines in $\mathscr{S}^{m,p}_{E,\mu}$. If E_b contains no points other than $\inf E$ and $\sup E$, we say the elements of $\mathscr{S}^{m,p}_{E,\mu}$ are *elementary splines*. In this case the set E without $\inf E$ and $\sup E$ is discrete and $\mu \leqslant m - 1$ on this set. If $a = \inf E > -\infty$, we may have $a \in E'$, hence $\mu(a) = m$, or $a \notin E'$ and $\mu(a) \leqslant m - 1$; similarly if $b = \sup E < +\infty$.

In Section 4 it was shown that every $\mathscr{S}^{m,p}_{E,\mu}$-spline extension F of a field $\{f\}$ breaks up into segments F_ν corresponding to the disjoint intervals J_ν in the decomposition $\mathbb{R} \setminus E_b = \bigcup J_\nu$. The $\mathscr{H}^{m,p}$-spline which interpolates the restriction of the field to $E_\nu = E \cap \bar{J}_\nu$ is an elementary spline and its restriction to $\bar{J}_\nu$ is F_ν. Thus we may restrict ourselves to consider only classes of elementary splines.

If $p = 2$ then $(D^m F)_s^{p-1} = D^m F$ and $\mathscr{S}$ is clearly a (linear) subspace of $\mathscr{H}^{m,2}$. For $p \neq 2$, $\mathscr{S}$ is not a subspace (nor convex), but is a cone since $F \in \mathscr{S}$ implies $\alpha F \in \mathscr{S}$ for every $\alpha \in \mathbb{R}$. We prove

THEOREM 6.1. *The cone $\mathscr{S} = \mathscr{S}^{m,p}_{E,\mu}$ of elementary splines is closed and nowhere dense in the normed space $\mathscr{H}^{m,p}$.*

Proof. That $\mathscr{S}$ is nowhere dense is trivial. In fact, choose an open interval J such that $J \cap E = \phi$. The restrictions of the functions of $\mathscr{S}$ to J are solutions of the differential equation $D^m(D^m F)_s^{p-1} = 0$ and clearly are nowhere dense in $\mathscr{H}^{m,p}(J)$. To show that $\mathscr{S}$ is closed, assume $F_n \in \mathscr{S}$ $(n = 1, 2,...)$, $F_n \to F$ in $\mathscr{H}^{m,p}$. Set

$$\begin{aligned} U_n &= (D^m F_n)_s^{p-1} = |D^m F_n|^{p-1} \operatorname{sgn}(D^m F_n), \qquad n = 1, 2,..., \\ U &= (D^m F)_s^{p-1} = |D^m F|^{p-1} \operatorname{sgn}(D^m F). \end{aligned} \tag{6.9}$$

The functions U_n and V are in $\mathscr{L}^q(\mathbb{R})$ ($p^{-1} + q^{-1} = 1$). The restriction of U_n to any bounded interval J such that $J \cap E = \phi$ is a polynomial of degree $\leqslant m - 1$, and the convergence $F_n \to F$ in $\mathscr{H}^{m,p}$ implies uniform convergence $U_n \to U$ on J, hence $U_n \to U$ in $\mathscr{L}^q(J)$. Therefore also $U_n \to U$ in $\mathscr{L}^q(K)$, where K is any compact interval such that $K \cap E$ is a finite set.

Given $\epsilon > 0$, let K_ϵ be so chosen that $K_\epsilon \cap E$ is a finite set, while $\int_{\mathbb{R}\setminus K_\epsilon} | D^m F |^p < (\frac{1}{2}\epsilon)^q$. Since $F_n \to F$ in $\mathscr{H}^{m,p}$ one can find N_ϵ such that $\int_{\mathbb{R}\setminus K_\epsilon} | D^m F_n |^p < (\frac{1}{2}\epsilon)^q$ for $n > N_\epsilon$. Then

$$\int_{\mathbb{R} \setminus K_\epsilon} | U |^q < (\tfrac{1}{2}\epsilon)^q, \qquad \int_{\mathbb{R} \setminus K_\epsilon} | U_n |^q < (\tfrac{1}{2}\epsilon)^q, \quad n > N_\epsilon$$

and since $U_n \to U$ in $\mathscr{L}^q(K_\epsilon)$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} | U_n - U |^q \leqslant \epsilon^q,$$

that is, $U_n \to U$ in $\mathscr{L}^q(\mathbb{R})$.

By (6.4),

$$\int_{\mathbb{R}} U_n D^m G = 0 \tag{6.10a}$$

for every function $G \in \mathscr{H}^{m,p}$ for which $D^k G(x) = 0$, $k = 0, 1,..., \mu(x) - 1$, $x \in E$. Therefore also

$$\int_{\mathbb{R}} U D^m G = 0 \tag{6.10b}$$

for every such function G. But this implies $F \in \mathscr{S}$, which was to be proved.

Suppose $F \in \mathscr{H}^{m,p}(\mathbb{R})$. F induces a Taylor field $\{f\}$ on E

$$f_k(x) = (1/k!)\, D^k F(x), \qquad k = 0, 1,..., \mu(x) - 1, \quad x \in E. \tag{6.11}$$

Let $F_* = S_{E,\mu}^{m,p} F = SF$ be the $\mathscr{H}^{m,p}$-spline interpolant of the field $\{f\}$, that is,

$$\begin{gathered} F_* \in \mathscr{S}_{E,\mu}^{m,p}, \\ D^k F_*(x) = D^k F(x), \qquad k = 0, 1,..., \mu(x) - 1, \quad x \in E. \end{gathered} \tag{6.12}$$

We consider F_* as an approximation in $\mathscr{S}_{E,\mu}^{m,p}$ to F, and the map $S: F \to F_* = SF$ as an approximation operator. If $p = 2$ then S is linear, and it is well known that S is the projection of $\mathscr{H}^{m,2}$ on $\mathscr{S}_{E,\mu}^{m,2}$, that is,

$$\| F - F_* \|_2 = \inf_{G \in \mathscr{S}} \| F - G \|_2, \tag{6.13}$$

where $\| \ \|_2$ is the norm (6.5) for $p = 2$. In fact,

$$\| F - G \|_2 = \| F - F_* - (G - F_*) \|_2 ,$$

and since $(G - F_*) \in \mathscr{S}$, while

$$D^k(F - F_*)(x) = 0 \qquad (k = 0, 1,..., \mu(x) - 1, x \in E),$$

we have by (6.4)

$$\int_{\mathbb{R}} D^m(G - F_*)\, D^m(F - F_*) = 0 \tag{6.14a}$$

and clearly also

$$\sum_{i=1}^{m} (G - F_*)(F - F_*)(x^i) = 0, \qquad i = 1,..., m. \tag{6.14b}$$

Thus

$$\| F - G \|_2^2 = \| F - F \|_2^2 + \| G - F_* \|_2^2 ,$$

and (6.13) is proved.

For $p \neq 2$, S is not linear, but hemilinear, i.e., $S(\alpha F) = \alpha SF$, $\alpha \in \mathbb{R}$. Nor is S the projection of $\mathscr{H}^{m,p}$ on $\mathscr{S}^{m,p}_{E,\mu}$ in the sense that $\| F - S \|_p = \inf_{G \in \mathscr{S}} \| F - G \|_p$. However,

$$S^2 = S \tag{6.15}$$

since the extremal $\mathscr{H}^{m,p}$-interpolant of $F_* = SF$ is clearly F_*. Moreover, S is of bound 1, that is

$$\| SF \|_p \leqslant \| F \|_p , \qquad F \in \mathscr{H}^{m,p}, \tag{6.16}$$

and equality holds in (6.16) if and only if $F \in \mathscr{S}$. This follows directly from the definition of S and Theorem 5.1, according to which the spline interpolant SF is the extremal $\mathscr{H}^{m,p}$-interpolant.

Beyond this we prove

THEOREM 6.2. *The approximation operator $S = S^{m,p}_{E,\mu}$ that maps the normed space $\mathscr{H}^{m,p}$ onto the cone $\mathscr{S} = \mathscr{S}^{m,p}_{E,\mu}$ of elementary $\mathscr{H}^{m,p}$-splines is hemilinear, idempotent, slightly continuous, and of bound* 1.

Proof. It remains to prove that S is slightly continuous. This means: $F_n \to F$ strongly in $\mathscr{H}^{m,p}$ implies $SF_n \rightharpoonup SF$ weakly in $\mathscr{H}^{m,p}$. Since $\| SF_n \|_p \leqslant \| F_n \|_p$, the sequence $\{SF_n\}$ is bounded, hence weakly compact.

Suppose $\{SF_\nu\}$ is a weakly convergent subsequence, $SF_\nu \rightharpoonup F_*$. Then $F_* \in \mathscr{H}^{m,p}$ and

$$D^k F_*(x) = \lim_{\nu\to\infty} D^k SF_\nu(x) = \lim_{\nu\to\infty} D^k F_\nu(x) = D^k(x),$$
$$k = 0, 1,..., \mu(x) - 1, x \in E. \tag{6.17}$$

Thus, F_* interpolates the same field $\{f\}$ that F does.

Set

$$U_\nu = (D^m SF_\nu)_s^{p-1}, \qquad U_* = (D^m F_*)_s^{p-1}. \tag{6.18}$$

The restriction of U_ν to any interval J which contains no point of E is a polynomial of degree $\leqslant m - 1$. On any compact interval K such that $K \cap E$ is a finite set $\{D^k U_\nu\}$ converges to $\{D^k U_*\}$ uniformly ($k = 0, 1, 2,...$). By Theorem 5.1 we conclude that F_* is the extremal $\mathscr{H}^{m,p}$-interpolant of the field $\{f\}$, thus $F_* = SF$ and $SF_\nu \rightharpoonup SF$. By familiar arguments one concludes that not only the subsequence $\{SF_\nu\}$, but the sequence $\{SF_n\}$ converges weakly to SF, which was to be proved.

Remark. Weak convergence of $\{SF_n\}$ implies uniform convergence of $\{D^k SF_n\}$ to $D^k SF$ for $k = 0, 1,..., m - 1$, on any compact subset of $\mathbb{R}$. By the above proof we also know that $\{D^m SF_n\}$ converges to $D^m SF$ uniformly on any compact interval K that contains only finitely many of the points of E. Hence also,

$$\lim_{n\to\infty} \int_K | D^m SF_n - D^m SF |^p = 0. \tag{6.19}$$

We have not been able to prove $\lim \| SF_n - SF \|_p = 0$. Of course, for $p = 2$ this is true, that is, $S_{E,\mu}^{m,2}$ is strongly continuous.

In the next theorem E is assumed to be one of the sequences (3.25a) or (3.25b), satisfying the conditions (3.29). It was proved in Theorem 3.2 that a Taylor field $\{f\}$ defined on such a sequence has an $\mathscr{H}^{m,p}$-extension if and only if

$$\sum_{x_i\in E} | f(x_i, x_{i+1},..., x_{i+m})|^p < \infty, \tag{6.20}$$

where the extended divided differences must be interpreted in the proper way. The class of all such fields is, in the obvious way, a linear space $\hbar = \hbar_{E,\mu}^{m,p}$ and we norm it as follows:

$$\{f\} \to \| \{f\} \|_p = \left\{ \sum_{i=1}^{m} | f_0(x^i)|^p + \sum_{x_i\in E} | f(x_i,..., x_{i+m})|^p \right\}^{1/p}. \tag{6.21}$$

Here $\{x^1, x^2,..., x^m\}$ is a subsequence of E, $x^1 \leqslant x^2 \leqslant \cdots \leqslant x^m$, and if $x^{i-1} < x^i = \cdots = x^{i+j} < x^{i+j+1}$ then $f_0(x^{i+k})$ in (6.21) stands for $f_k(x^{i+k})$ $(k = 0,..., j)$. It is readily seen that $\mathscr{k}_{E,\mu}^{m,p}$ with the above norm is a uniformly convex Banach space.

We define the map $R = R_{E,\mu}^{m,p}$ from the space $\mathscr{k}$ onto the cone $\mathscr{S}$ by

$$R_{E,\mu}^{m,p}\{f\} = F_* = S_{E,\mu}^{m,p}\{f\}. \tag{6.22}$$

We consider $\mathscr{S}$ both with the relative strong and weak topologies of the space $\mathscr{H}^{m,p}$. In the context of the boundary-value problem, R is the map of the boundary-value vector as an element of $\mathscr{k}$ onto the solution as an element of $\mathscr{S}$; R^{-1} is the inverse map. We prove

THEOREM 6.3. *The map $R = R_{E,\mu}^{m,p}\colon \mathscr{k}_{E,\mu}^{m,p} \to \mathscr{S}_{E,\mu}^{m,p}$ is bounded and slightly continuous. The inverse map R^{-1} is bounded and continuous.*

Proof. For $F_* \in \mathscr{S}$ set $\{f\} = R^{-1}F_*$. In the first part of the proof of Theorem 3.1 (with extension indicated in the proof of Theorem 3.1a) it is shown that

$$\sum_{x_i \in E} |f(x_i, x_{i+1},..., x_{i+m})|^p \leqslant C_1 \int_{\mathbb{R}} |D^m F_*|^p \tag{6.23}$$

for some constant C_1. On the other hand, $\sum_{i=1}^m |f(x^i)|^p = \sum_{i=1}^m |F_*(x^i)|^p$. Thus R^{-1} is bounded. In similar way one finds for $F_*^n \in \mathscr{S}$, $\{f^n\} = R^{-1}F_*^n$

$$\sum_{x_i \in E} |f(x_i,..., x_{i+m}) - f^n(x_i,..., x_{i+m})|^p \leqslant C_2 \int_R |D^m F_* - D^m F_*^n|^p \tag{6.24}$$

for some constant C_2. One concludes that R^{-1} is continuous.

For $\{f\} \in \mathscr{k}$ set $F_* = R\{f\}$. By the second part of Theorem 3.1 we have

$$\int_{\mathbb{R}} |D^m F_*|^p \leqslant C_3 \sum_{x_i \in E} |f(x_i,..., x_{i+m})|^p \tag{6.25}$$

for some constant C_3. Hence R is bounded. Since $\mathscr{S}$ with the weak topology is locally compact, R is bounded and R^{-1} is continuous one concludes in familiar fashion that R, as a map from $\mathscr{k}$ to $\mathscr{S}$ with the weak topology, is continuous. Thus the theorem is proved.

Remark. As before, we observe that weak convergence of $\{F_*^n\}$ implies uniform convergence of $\{D^k F_*^n\}$ for $k = 0, 1,..., m-1$ on any compact subset of $\mathbb{R}$; also $\lim \int_K |D^m F_*^n - D^m F_*|^p = 0$ for any compact interval K that contains only finitely many points of E. For $p = 2$, the map $R = R_{E,\mu}^{m,2}$ is linear, hence boundedness of R implies (strong) continuity. In this case, R is a norm isomorphism between $\mathscr{k}$ and $\mathscr{S}$ as a subspace of $\mathscr{H}^{m,2}$.

7. Cardinal Interpolation

The problem of cardinal interpolation is to determine a function F, with specified properties, that interpolates (extends) a given function f whose domain is $\mathbb{Z}$, the class of rational integers. We also include the problem of interpolating a Taylor field $\{f\}$ with domain $\mathbb{Z}$, where we require so-called Hermite interpolation of the data

$$(1/k!)\, D^k F(i) = f_k(i), \qquad k = 0, 1,..., \mu_i - 1, \quad i \in \mathbb{Z}. \tag{7.1}$$

μ_i is the multiplicity of the interpolation node i.

By Theorem 3.2 there exists a solution $F \in \mathscr{H}^{m,p}(\mathbb{R})$ of this last problem with $\mu_i \leqslant m$ ($i \in \mathbb{Z}$) if and only if

$$\sum_{i=-\infty}^{\infty} \sum_{k=0}^{\mu_i - 1} | \Delta_k{}^m f(i)|^p < \infty. \tag{7.2}$$

Here $(1/m!)\, \Delta_k{}^m f(i)$ denotes the extended divided difference $f(x_1, x_2, ..., x_{m+1})$ of Section 3, where

$$x_1 = x_2 = \cdots = x_{\mu_i - k} = i, \qquad x_{\mu_i - k+1} = \cdots = x_{\mu_i - k + \mu_{i+1}} = i + 1,$$

etc. If (7.2) is satisfied there is, by Theorem 5.1, a unique $\mathscr{H}^{m,p}$-spline F_* of the field $\{f\}$, and F_* is the extension of $\{f\}$ with minimal $\int_{\mathbb{R}} | D^m F|^p$. If $p = 2$, F_* is the common polynomial (of degree $2m - 1$) spline with $2m - 1 - \mu_i$ continuous derivatives at the point $i \in \mathbb{Z}$. The above result generalizes Theorem 2 of Schoenberg's paper [10], where $p = 2$, $\mu = 1$ are assumed (also compare Theorem 9 of [11]).

We wish to present an $\mathscr{H}^{m,p}$-interpolant of the cardinal field (7.1), which is not the extremal interpolant (unless $p = 2$), but has the advantage that it is given as a series expansion involving a finite (small) number of basic functions rather than the solution of a boundary-value problem with infinitely many boundary conditions. For simplicity we consider only the case

$$\mu_i = \mu = \text{const}, \qquad 1 \leqslant \mu \leqslant m \qquad i \in \mathbb{Z}. \tag{7.3}$$

In this case we have for $k = 0, 1,..., \mu - 1$

$$\frac{1}{m!} \Delta_k{}^m f(i)$$
$$= [i, i,..., i, i + 1, i + 1,...; f_0(i), f_1(i),..., f_{\mu-k-1}, f_0(i + 1), f_1(i + 1),...] \tag{7.4}$$

in the notation of Section 3, with i appearing $(\mu - k)$ times, $(i + 1)$ μ times (if $2\mu - k \leqslant m + 1$), etc.

Consider, in particular, fields $\{e^k\}$ $(k = 0, 1,..., \mu - 1)$ defined such that

$$\Delta_j{}^m e^k(i) = \delta_{i0}\delta_{jk}, \qquad j = 0, 1,..., m - 1, \quad i \in \mathbb{Z}. \tag{7.5}$$

Let G^k be the function G of the proof of Theorem 3.2 [see (3.39)] for the configuration X^k [see (3.34)], where the first $\mu - k$ x's are 0, the next μ (if $2\mu - k \leqslant m + 1$) are 1, etc. Then we have a function $G^k \in \mathscr{H}^{m,p}$ such that $e^k = G^k$ satisfies (7.5). Let H^k be the $\mathscr{H}^{m,p}$-spline that interpolates the field $\{G^k\}$ on $\mathbb{Z}$ so that

$$\Delta_j{}^m H^k(i) = \delta_{i0}\delta_{jk}, \qquad j = 0, 1,..., m - 1, \quad i \in \mathbb{Z}. \tag{7.6}$$

$H^k + P$, for any $P \in \mathscr{P}^{m-1}$ would also be such a spline. The particular choice of G^k serves to eliminate the ambiguity in the definition of H^k. Since $G^k(x) = 0$ for $x < 0$ [see (3.39)], we also have

$$D^k H^k(i) = 0, \qquad k = 0, 1,..., \mu - 1; \quad i = -1, -2,... . \tag{7.7}$$

The series

$$F_+(x) = \sum_{i=0}^{\infty} \sum_{k=0}^{\mu-1} \Delta_k{}^m f(i)\, H^k(x - i) \tag{7.8a}$$

converges for $x = -1, -2,...$ [where $F_+(x) = 0$], and

$$\left\{\int_{\mathbb{R}} \left| D^m \sum_{i=N}^{N+n} \sum_{k=0}^{\mu-1} \Delta_k{}^m f(i)\, H^k(x - i)\right|^p dx\right\}^{1/p}$$

$$\leqslant \sum_{i=N}^{N+n} \sum_{k=0}^{\mu-1} \left\{| \Delta_k{}^m f(i)|^p \int_{\mathbb{R}} | D^m H^k |^p\right\}^{1/p}.$$

Because of hypothesis (7.2) it follows that (7.8) converges in the normed space $\mathscr{H}^{m,p}(\mathbb{R})$. The same is true of the series

$$F_-(x) = \sum_{i=-\infty}^{-1} \sum_{k=0}^{\mu-1} \Delta_k{}^m f(i)\, H^k(-x + i). \tag{7.8b}$$

Because of (7.6) we have

$$\Delta_k{}^m(F_+ + F_-)(i) = \Delta_k{}^m f(i), \qquad k = 0, 1,..., \mu - 1, \quad i \in \mathbb{Z}. \tag{7.9}$$

Finally, we determine $P \in \mathscr{P}^{m-1}$ such that

$$D^k P(0) = \left(\frac{1}{k!} f_k - F_+\right)(0), \qquad k = 0, 1,..., \mu - 1,$$
$$D^k P(1) = \left(\frac{1}{k!} f_k - F_+\right)(1), \qquad k = 0, 1,..., \mu - 1, \tag{7.10}$$

etc. Then

$$F = P + F_+ + F_- \tag{7.11}$$

is a function in $\mathscr{H}^{m,p}$ which satisfies (7.1).

In the case $p = 2$, (7.11), being the sum of $\mathscr{H}^{m,2}$-splines, is an $\mathscr{H}^{m,2}$-spline. In this case the functions

$$1, x,..., x^{m-1}, H^k(x), H^k(x-1),..., H^k(-x+1), H^k(-x+2),..., \quad k = 0, 1,..., \mu - 1 \tag{7.12}$$

form a Schauder basis of the Hilbert space $\mathscr{H}^{m,2}_{\mathbb{Z},\mu}$ (see Section 6).

THEOREM 7.1. *The functions* $H^k(x-i)$, $H^k(-x+i)$ *defined above* ($k = 0, 1,..., \mu - 1$; $i = 0, 1, 2,...$) *are* $\mathscr{H}^{m,p}$*-splines, and for any Taylor field* $\{f\}$ *on* $\mathbb{Z}$ *of constant height* $\mu \leqslant m$ *that can be interpolated by an* $\mathscr{H}^{m,p}$*-function the expansion*

$$F(x) = P(x) + \sum_{k=0}^{\mu-1} \left\{ \sum_{i=-\infty}^{-1} \Delta_k{}^m f(i)\, H^k(-x+i) + \sum_{i=0}^{\infty} \Delta_k{}^m f(i)\, H^k(x-i) \right\},$$

with $P \in \mathscr{P}^{m-1}$ *chosen according to* (7.10), *converges in the normed space* $\mathscr{H}^{m,p}(\mathbb{R})$ *and is an* $\mathscr{H}^{m,p}$*-interpolant of* $\{f\}$. *If* $p = 2$ *then* F *is the extremal* $\mathscr{H}^{m,2}$*-interpolant of* $\{f\}$, *and the functions* (7.12) *form a Schauder basis of the Hilbert space* $\mathscr{H}^{m,2}_{\mathbb{Z},\mu}$.

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